Barrier Option Pricing Using Adjusted Transition Probabilities

Abstract

In the existing literature on barrier options, much effort has been exerted to ensure convergence through placing the barrier in close proximity to, or directly onto, the nodes of the tree lattice. In this paper we show that this may not be necessary to achieve accurate option price approximations. Using the Cox/Ross/Rubinstein binomial tree model and a suitable transition probability adjustment we demonstrate that our "probability-adjusted" model exhibits increased convergence to the analytical option price. We study the convergence properties of various types of options including (but not limited to) double knock-out, exponential barrier, double (constant) linear barriers and linear time-varying barriers. For options whose strike price is close to the barrier we are able to obtain numerical results where other models fail and, although convergence tends to be slow, we are able to calculate reasonable approximations to the analytical option price without having to reposition the lattice nodes.

Keywords: barrier option, binomial tree, convergence rate, transition probability

1. Introduction

Methods for pricing barrier options consist of two approaches: numerical based methods and theoretical closed-form expressions. In this paper we use the former approach with a suitable transition probability adjustment to demonstrate an increased convergence rate for the standard Cox-Ross Rubinstein (1979) binomial tree model applied to barrier options.

For European-style exercise put and call options the Cox-Ross-Rubinstein (CRR) binomial tree model is able to yield convergence towards the "true" (i.e. continuous time model) option price. In the case of a plain-vanilla option, convergence of the standard binomial lattice to the analytic value generally occurs within a few hundred time steps, yet typically a persistent bias in the price of the option remains. However, if the CRR method is used to price more complex options such as single (constant) barrier, multiple barrier and time-varying barrier options, the CRR method converges so slowly and exhibits such large bias that its use becomes impractical. This is an important observation because, due to their lower cost and popularity in hedging financial and commodity positions, barrier and path-dependent options are now commonplace across all financial markets. Consequently, a need for improved lattice pricing models exists, particularly for the more complex options.

To explain the upward bias that occurs when pricing options using the CRR method, Boyle and Lau (1994) studied what happens to the price of a given option when the distance of the barrier to a layer of nodes in the binomial tree varies. They found that the upward bias problem arises as a consequence of the discretization of the nodes of the binomial lattice. Specifically, the bias obtains when the option barrier passes between two successive layers of nodes comprising the binomial tree without coming close to a node. Numerically, this is perceived as a mispricing of the option that takes the form of an upward bias (i.e. convergence to the "true" price from above) added to the option price. In order to reduce this bias, Boyle and Lau (1994) proposed to reposition the nodes in the binomial lattice such that the barrier passes as close as possible to a given layer of nodes in the binomial tree. With this modification, increased convergence rates were achieved; in one case Boyle and Lau report that convergence improved from 800 steps in the binomial tree to just 21 steps. This variation in the distance of the barrier from a layer of nodes manifests itself as an observable pattern in a plot of the option price against the number of time steps of the binomial lattice. One observes a series of alternating crests and troughs in a plot of the option price versus the number of time steps in the tree. Accurate approximations to the "true" option price occur at the troughs of the convergence graph where the barrier lies in close proximity to a given layer of the binomial lattice. Conversely, relatively poor price approximations occur at the crests.

It is known that the node-repositioning technique of Boyle and Lau is unable to produce an approximation to the option price when the initial underlying price lies close to the barrier, when there are multiple barriers or when the barrier is time-varying. Ritchken (1995) addresses these particular problems by employing the use of a trinomial lattice based upon the multinomial model of Kamrad and Ritchken (1991). Ritchken's trinomial tree based method is also based upon a repositioning of the lattice nodes, however it differs from the method of Boyle and Lau in that Ritchken positions the lattice using a stretch-factor so that the barrier lies exactly upon a given level of the lattice nodes. Thus, despite increasing the number of time-divisions of the tree, there always exists a layer of nodes that coincides with the barrier level resulting in a rapid convergence to the "true" option price. While exhibiting good convergence, Ritchken's method suffers from two drawbacks. The first is that his method encounters difficulty converging to the "true" option price (and in some cases fails to converge at all) when the initial underlying price is very close to the barrier. The second drawback is that if a parameter of the option changes (maturity, volatility, etc…) then the entire lattice must be repositioned before calculating the new option price.

Thus, option pricing using lattice techniques becomes a trade-off between convergence rates and having to reposition the nodes of the lattice in proximity to the barrier. In the following work, we propose a simple modification of the CRR binomial model, specifically an adjustment to the transition probabilities of the CRR binomial tree that eliminates the need to reposition the nodes of the lattice. Simultaneously, this approach yields significantly increased convergence rates compared to the CRR model. To calculate the "true" option price we make use of a series of analytical methods and empirical results available in the barrier option pricing literature. We are thus able to compare our calculations to a known value.

The remainder of the paper is organized as follows. Section 2 briefly reviews the CRR model of the binomial tree in order to make the paper self-contained and discusses the adjustment to the transition probabilities of the CRR binomial tree. In section 3, we present the results of our convergence rate analysis and propose a general method designed to improve the determination of the "true" option price from our calculations. Section 4 contains our concluding remarks.

2. The Probability-Adjusted Cox-Ross-Rubinstein Binomial Tree Model

The binomial tree model of Cox, Ross and Rubinstein (1979) has long been a mainstay in the empirical option pricing literature and is a highly flexible discrete-time option pricing technique that has spawned a large field of research. The basic CRR model uses the quantities u for an upward move and d for a downward move in the price of the underlying asset in order to construct the binomial tree for the underlying asset[1]. Thus an upward move in the price of the underlying gives rise to a new price uS in the subsequent time period and a downward move in the price of the underlying gives rise to a price in the next period of dS. Using the notation of Hull (2003) we have that

$$u = \exp\left(\sigma\sqrt{\delta t}\right) \tag{1}$$

$$d = \exp\left(-\sigma\sqrt{\delta t}\right) = \frac{1}{u} \tag{2}$$

where σ is the volatility and δt is the size of the time-step of the tree. A simple binomial tree is illustrated in Figure 1.

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[1] In this work, the binomial tree that we construct for the underlying asset is the same regardless of the type of option being priced. The various types of options that we treat in this work are the following: down-and-out call option, up-and-out call option, down-and-out option with a linear time-varying barrier, a double constant barrier option, down-and-out with exponential barrier, up-and-out put option and finally a double time-dependent linear barrier option. Only the option-specific tree varies between the diverse number of options that we consider.

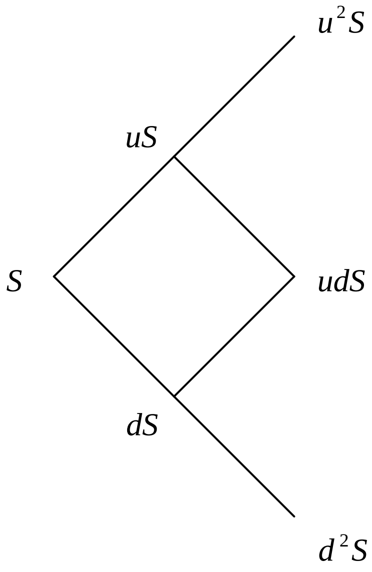

Figure 1: The two-step binomial tree for the underlying asset.

The underlying binomial tree is used to construct the option-pricing tree. To do this, one utilizes the risk-neutral transition probabilities to calculate the price of the option at a given time-step (i.e. node) of the tree. Subsequently, the option tree is constructed by working backwards through the lattice until an approximation to the "true" option price is obtained. This requires use of the risk-neutral transition probabilities. Given a risk-free rate r they are given by

$$p_{up} = \frac{\exp(r\delta t) - d}{u - d} \tag{3}$$

$$p_{down} = 1 - p_{up} \tag{4}$$

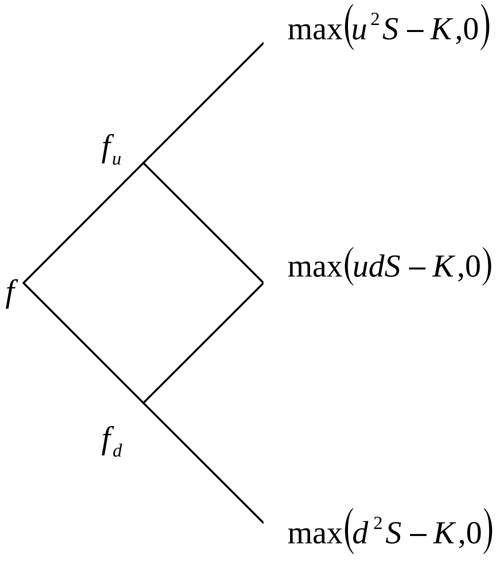

Figure 2: Construction of the option tree associated with the underlying tree in Figure 1.

Option prices at each node of the tree are calculated in the following manner. Using the values of the option price in the next time period, the approximation to the "true" option price at a given node is calculated in the current period as the discounted sum of the two option prices achievable by moving from the current node to the node in the next time period. These prices are weighted by the probabilities p_{up} and p_{down} for an up and down step, respectively, and the entire expression is discounted at the prevailing constant interest rate. Using the sample tree given in Figure 2, the approximation to the option price, f, would be calculated as

$$f = \left(p_{up} f_u + p_{down} f_d\right)\exp(-r\delta t) \tag{5}$$

where $p_{down} = 1 - p_{up}$.

Iterating (backwards) through the binomial option tree in this manner, the final calculation produces an estimate of the option price. Naturally, the greater the number of time divisions of the tree, the greater the accuracy of the approximation to the true price.

A *barrier option* is a path-dependent option whose payoff is determined based on whether the price of the underlying asset reaches some pre-determined level. For example, in the case of a down-and-out[2] barrier call option, the option payoff is set to zero when the underlying price is below the barrier. A barrier option can be priced using the same binomial tree method employed to price standard European call options, however the binomial tree will converge extremely slowly to the price of the option. As discussed, it is possible to reposition the nodes of the lattice to increase convergence, but there is another relatively intuitive way to effect a greater convergence rate. This idea has been used previously to increase the convergence rates of Monte Carlo option pricing algorithms. It is known that, due to the discretized nature in which the underlying asset price evolves, it is possible for the underlying asset price to breach the option barrier without being detected by the Monte Carlo simulation; see for example, Geman and Yor (1996). One way to alleviate this problem is to use the supremum of a Brownian bridge to calculate the probability that the underlying asset price touches the barrier for any given step of the simulation. Such a method was utilised by Baldi (1995). As noted in Baldi et al. (1999), this method is not without its limitations in that it cannot be effectively used to price multiple barrier options. Thus, Baldi et al. (1999) derive a series of approximations for the exit probability of the Brownian bridge that can be used to effectively price multiple barrier and time-varying barrier options. Baldi et al. use these probabilities to improve upon the Monte Carlo calculations whereas our contribution is to

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[2] In the case of a down-and-in call option, the value of the call option is set to zero if the underlying asset price *does not* touch the barrier.

demonstrate that these probability estimates can also be used to accelerate the convergence rate of a binomial tree. Our procedure is described in the following sections.

2.1 Down-and-out Call Option

As an introductory example, consider the three-period underlying asset binomial tree for a down-and-out call option illustrated in Figure 3. In the figure, L is the (lower) barrier of the option, and the indexed values of S represent the node prices for the various time periods indexed by the time division, $\delta t = T/n$, T being the option maturity and n the number of divisions of the tree. We construct the binomial trees for the underlying asset and the option in the usual fashion with one difference: we modify the transition probability of the binomial tree if we detect a potential barrier crossing between the nodes in the current time period and the following time period. In Figure 3, such a situation would arise in the transition from S_{T_0} to $S_{T_{0+\delta t}}$. Thus, we need to adjust the related transition probability. To perform the adjustment we multiply the relevant transition probability by the appropriate exit probability, in this case that given by Baldi et al. (1999):

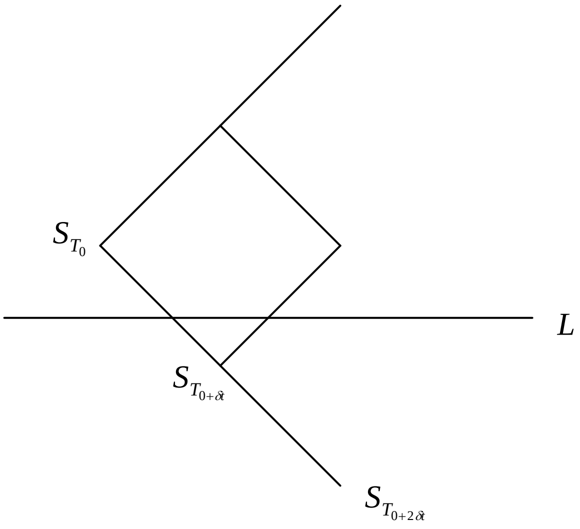

Figure 3: The binomial option price tree for a down-and-out call option with constant linear barrier B_1.

$$p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left[-\frac{2}{\sigma^2 \delta t}\right]\ln\left(\frac{S_{T_0}}{L}\right)\ln\left(\frac{S_{T_{0+\delta t}}}{L}\right) \tag{6}$$

Consequently, with this probability adjustment, the probability adjusted price of the option can be written as

$$C_{DAO} = \exp(-r\delta t)p_{up}\left(1 - p_L^{\delta t}\right)C\left(S_{T_{0+\delta t}}^{up}\right) \tag{7}$$

where C_{DAO} is the price of the down-and-out call option and $C\left(S^{up}_{T0+\delta t}\right)$ is the price of the call option at node $S^{up}_{T0+\delta t}$. Our probability adjustment is equivalent to changing the binomial tree into a trinomial tree in the neighbourhood of the barrier. The third branch reflects the probability that the barrier is reached at an intermediate time. The option is then cancelled. Therefore the third branch does not contribute to the option value and it can be omitted.

2.2 Up-and-Out Call Option

The situation for the up-and-out call option differs very little from that of the down-and-out call.[3] We construct the binomial and option trees for a regular up-and-out call option, but use the following exit probability correction given in Baldi et al. (1999):

$$p_U^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left[-\frac{2}{\sigma^2 \delta t}\right] \ln\left(\frac{U}{S_{T_0}}\right) \ln\left(\frac{U}{S_{T_{0+\delta t}}}\right) \tag{8}$$

Thus, the option is priced according to:

$$C_{UAO} = \exp(-r\delta t) p_{down}\left(1 - p_U^{\delta t}\right) C\left(S^{down}_{T0+\delta t}\right) \tag{9}$$

2.3 Down-and-Out Call Option With a Linear, Time-Dependent Barrier

As a final example, this section of the paper treats the situation of an option with a time-dependent, linear barrier. In this case, we can define the barrier using a linear equation consisting of a slope and an intercept:

$$L_t = l_1 t + l_0 \tag{10}$$

where l_1 and l_0 are the slope and intercept, respectively. With a time-dependent barrier, we can draw a picture similar to that for the down-and-out call with a constant barrier, see Figure 4.

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[3] In this case, we omit the diagram for the up-and-out call. The situation is analogous to the down-and-out call option only the option value is zero when the underlying price breaches the barrier from below.

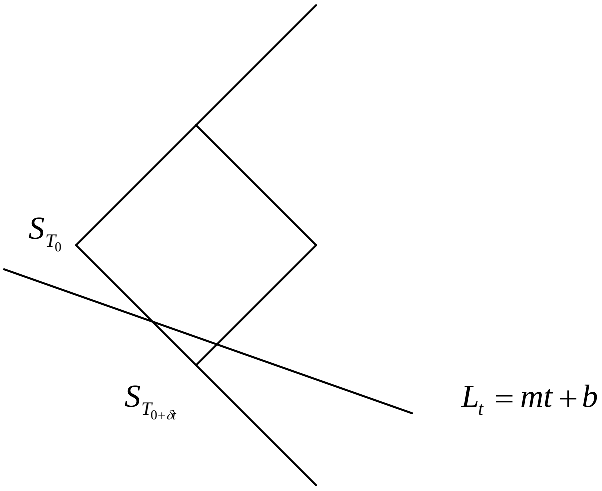

Figure 4: binomial option price tree for a down-and-out call option with a time-varying, linear barrier described by equation (10).

The appropriate exit probability in this case is given as (Baldi et al. (1999)):

$$p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2}\ln\left(\frac{S_{T_0}}{L_{T_0}}\right)\left[\frac{1}{\delta t}\ln\left(\frac{S_{T_{0+\delta t}}}{L_{T_0}}\right) + \frac{m}{b + m\delta t}\right]\right\} \tag{11}$$

In a similar fashion to the previous cases, we correct the transition probability in the following manner. We price the underlying using the conventional binomial tree. At each node in the tree, we also calculate the level of the barrier associated with a particular node using the equation for the barrier, in this case equation (10). Subsequently, we work backwards through the option tree and check to see if, between the current node in the tree and a node in the "next" period[4] of the tree, there is a potential crossing of the barrier. Thus, there exists the probability that the underlying price has traversed the barrier so we need to account for the exit probability. Consequently, we calculate the transition probability adjustment using equation (11). This results in a new "adjusted" transition probability between the relevant nodes of the binomial tree that corrects for the probability of barrier exit.

It is possible to write similar expressions for a down-and-out call option with an exponential time-dependent barrier, an up-and-out put option and finally a double linear time-dependent barrier option. Because the pricing procedure is similar in all cases, in order to avoid unnecessary repetition, we summarize the probability adjustments of Baldi et al. (1999) used in this paper in Tables 1 and 2.

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[4] In actuality, this is the *previous* time-period, but recall that we are working *backwards* through the tree.

Option Type	Probability Adjustment
Down-and-out call	$p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left[-\frac{2}{\sigma^2 \delta t}\right]\ln\left(\frac{S_{T_0}}{L}\right)\ln\left(\frac{S_{T_{0+\delta t}}}{L}\right)$
Up-and-out call	$p_U^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left[-\frac{2}{\sigma^2 \delta t}\right]\ln\left(\frac{U}{S_{T_0}}\right)\ln\left(\frac{U}{S_{T_{0+\delta t}}}\right)$
Down-and-out call with time-dependent linear barrier	$p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2}\ln\left(\frac{S_{T_0}}{L_{T_0}}\right)\left[\frac{1}{\delta t}\ln\left(\frac{S_{T_{0+\delta t}}}{L_{T_0}}\right) + \frac{m}{b + m\delta t}\right]\right\}$
Down-and-out call with exponential barrier	$p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2}\left(\ln\left(S_{T_0}\right) - L_1 - L_2 T_0\right)\left(\frac{\ln\left(S_{T_{0+\delta t}}\right) - L_1 - L_2 T}{\delta t} - L_2\right)\right\}$

Table 1: Single barrier option type and associated probability adjustment equation.

Option Type	Probability Adjustment
Down-and-out call with double constant linear barriers	$p_U^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2 \delta t}\ln\frac{U}{S_{T_0}}\ln\frac{U}{S_{T_{0+\delta t}}}\right\}$ $p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2 \delta t}\ln\frac{S_{T_0}}{L}\ln\frac{S_{T_{0+\delta t}}}{L}\right\}$
Double time-dependent linear barrier option	$p_U^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2}\ln\left(\frac{U}{S_{T_0}}\right)\left[\frac{1}{\delta t}\ln\left(\frac{U}{S_{T_{0+\delta t}}}\right) - \frac{m_{upper}}{b_{upper} + m_{upper}\delta t}\right]\right\}$ $p_L^{\delta t}\left(T_0, S_{T_0}, S_{T_{0+\delta t}}\right) = \exp\left\{-\frac{2}{\sigma^2}\ln\left(\frac{S_{T_0}}{L}\right)\left[\frac{1}{\delta t}\ln\left(\frac{S_{T_{0+\delta t}}}{L}\right) + \frac{m_{lower}}{b_{lower} + m_{lower}\delta t}\right]\right\}$

Table 2: Multiple barrier option type and associated probability adjustment equation.

In section 3, we use these probabilities to calculate our option price approximations.

3. Results

3.1 Down-and-out Call with Single Constant Barrier

To gauge the effectiveness of our model against those in the literature, we priced several different types of barrier options using our probability adjustment method and compared the results of these calculations to various results published in the literature.

Ritchken (1995) presents many numerical results for both binomial and trinomial tree scenarios. We generated results for the case of a down-and-out call option with parameters

$S_0 = 95$, $K = 100$, $\sigma = 25\%$, $r = 10\%$, $T = 1$, and a constant barrier level of 90. These parameter values coincide with those used by Ritchken (1995). The results of our calculations compared to the values obtained by other authors are presented in Table 3. The first observation we can make is that the probability adjustment technique yields option price values that lie between those produced by the methods of Boyle and Lau and Ritchken. The reason for this can be understood by considering how each method prices the option. The standard binomial tree method is a relatively inaccurate method by which to price a barrier option. The intuition for this is that, generally, the barrier does not consistently pass in close proximity to the lattice nodes. Consequently, the barrier-to-node distance in not always minimal and the approximation to the analytic option price is therefore not optimal. It thus requires a large number of tree levels to produce an accurate approximation. In direct contrast, Ritchken's trinomial tree method with lattice-stretching parameter, λ, works on the basis that the nodes of the trinomial tree lattice are repositioned, or "stretched", such that the barrier consistently passes through a node for a given level of the tree. This ensures a highly accurate approximation to the analytic option price. However, as we shall see, there are circumstances under which this method breaks down. In contrast, our method utilises an adjustment to the binomial tree transition probability in order to prevent the price approximation from diverting significantly from the analytical price. That is, if there is a probability that the barrier will be crossed during the transition from one layer of the binomial tree to the next, the probability adjustment corrects for the

Number of Time Steps in Tree	**Standard Binomial Tree (Boyle and Lau)**	**Trinomial Tree (Ritchken)**	**Option Price (probability adjustment)**
25	8.8486	6.0069	6.0385
50	7.2405	5.9942	6.3882
75	6.3001	5.9899	6.1425
100	7.5045	5.977	6.0465
150	6.5612	5.9976	6.2312
200	7.2307	5.9986	6.0011
400	6.6505	5.9977	6.0783
800	6.6040	5.9974	6.0065
1000	6.1002	5.9972	6.0458
2000	6.1449	5.9970	6.0591
3000	6.0542	5.9969	6.0233
4000	6.0990	5.9969	6.0404
Analytical Value	5.9968	5.9968	5.9968

Table 3: Comparison of the option price calculated with the adjusted transition probability method and the results of both Boyle and Lau (1994) and Ritchken (1995). Parameters used in the calculation are: S_0=95, K = 100, σ = 25%, T = 1 year, r = 10%, barrier = 90.

potential crossing of the barrier thereby fixing the subsequent node price close to the level of the barrier while not necessarily resulting in the barrier touching the node. However, this probability adjustment does not ensure that the barrier and node level coincide. The effect of this correction will be more evident when we consider the convergence properties of our method compared to those of a standard binomial tree. The overall result is a faster convergence to the analytical value of the option compared to the regular binomial method, but with the presence of a small upward bias in the price approximation.

Continuing with this particular scenario, we can compare our results to those of the “revised” binomial and trinomial trees of Boyle and Lau (1994) and Ritchken (1995), respectively. Ritchken presents results calculated for tree levels that satisfy the condition that the lattice produces a layer of nodes that lie directly on the barrier. This is given by Ritchken (1995) as the non-integer value η, that satisfies

$$\eta = n_0\lambda \tag{12}$$

where n_0 is the required number of consecutive downward moves that leads to the lowest layer of nodes above the barrier level and λ is the stretch parameter. The comparison of empirical results is shown in Table 4.

Number of Tree Levels (node value)	**Revised Binomial (Boyle and Lau, 1994)**	**“Stretched” Trinomial (Ritchken, 1995)**	**Probability Adj.**
85	6.020	5.9889	5.9489
192	6.006	5.9967	5.9576
342	5.998	5.9973	5.9647
534	6.000	5.9973	5.9706

Table 4: Summary of option price approximations for node values satisfying $\eta=n_0\lambda$.

Of course, due to the inherent nature of the adjusted probability method[5], there is no correlation between these particular tree level values and the accuracy of our option price approximation. We do not, therefore, expect our best approximations to the option price to coincide with these node values.

In a final comparison, we study the behaviour of the probability adjustment method when the stock price approaches the barrier. In these circumstances, many lattice-based option valuation techniques have difficulty producing an approximation to the option price. In the case of the standard binomial tree, convergence is so slow that even after 5000 time divisions, there is significant difference between the approximate and analytical values. Even Ritchken’s “stretched”

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[5] We are not using a node repositioning technique.

trinomial tree method encounters difficulty in pricing a down-and-out call when the stock price approaches the barrier. For example, in Table 5, Ritchken's method first encounters difficulty at 500 time-divisions and a stock price of 91.0. Furthermore, when the initial stock price is very close to the barrier, even as many as 5000 iterations are unable to provide an approximation to the option price. The reason for this is that in order for Ritchken's trinomial tree to construct the lattice, one must be able to calculate a value η, given by

$$\eta = \frac{\ln\left(s(0)/b\right)}{\sigma\sqrt{\Delta t}} \tag{13}$$

In some instances (e.g. when the stock price is close to the barrier), it is not possible to find an integer smaller than η and, consequently, Ritchken's method in unable to price the option. Conversely, because the probability adjustment technique is based on the binomial tree method with no node repositioning, we are always able to produce an approximation to the option price regardless of the stock price to barrier distance. Table 5 presents our option price approximations in comparison to those of Ritchken's as the stock price approaches the barrier. At a stock value of 90.3, Ritchken's method is unable to produce an approximation to the price in 5000 divisions or less. Conversely, we obtain an approximation with 500 time-divisions[6]. Furthermore, we are able to obtain price approximations when the stock price is extremely close to the barrier. Additionally, the quality of the approximation improves with decreasing distance between the stock price and barrier and, in the extreme case of a stock price of 90.01, we have very good agreement between our approximation and the analytical price.

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[6] In actuality, we can produce an estimate for *any number* of time divisions of the tree.

	Number of Time Steps in the Tree						
Stock Price	**500**	**1000**	**2000**	**3000**	**4000**	**5000**	**Analytic Price**
94.0	4.910 (4.863)	4.957 (4.864)	4.915 (4.864)	4.920 (4.864)	4.852 (4.864)	4.886 (4.864)	**4.864**
93.0	3.720 (3.700)	3.716 (3.701)	3.733 (3.702)	3.715 (3.701)	3.728 (3.701)	3.722 (3.702)	**3.702**
92.0	2.500 (2.504)	2.589 (2.506)	2.515 (2.506)	2.546 (2.506)	2.563 (2.506)	2.521 (2.506)	**2.506**
91.5	2.047 (1.894)	1.901 (1.894)	1.894 (1.895)	1.963 (1.895)	1.907 (1.895)	1.945 (1.895)	**1.895**
91.0	1.242 -	1.365 (1.274)	1.263 (1.274)	1.331 (1.275)	1.315 (1.275)	1.279 (1.274)	**1.274**
90.5	0.810 -	0.758 -	0.624 -	0.663 -	0.691 (0.642)	0.699 (0.642)	**0.642**
90.4	0.649 -	0.642 -	0.576 -	0.508 -	0.521 -	0.537 (0.515)	**0.515**
90.3	0.476 -	0.490 -	0.479 -	0.450 -	0.419 -	0.390 -	**0.387**
90.2	0.303 -	0.316 -	0.327 -	0.328 -	0.323 -	0.316 -	**0.258**
90.1	0.142 -	0.146 -	0.152 -	0.156 -	0.159 -	0.161 -	**0.129**
90.05	0.068 -	0.069 -	0.071 -	0.072 -	0.073 -	0.074 -	**0.065**
90.01	0.013 -	0.013 -	0.013 -	0.013 -	0.013 -	0.013 -	**0.013**

Table 5: This table presents the results for the behaviour of the down-and-out call option price when the stock price is close to the barrier. The last column contains the price of the down-and-out call option using an analytical solution. Prices in brackets are those given in Ritchken (1995). Prices indicated by “-” were unable to be computed using the node repositioning method of Ritchken - that is the number of partitions used were insufficient to produce an approximation to the analytical price.

3.2 Double Knock-Out European Call Option

Before considering the time-varying barrier cases, we compare our results to those of Ritchken (1995) for the case of a double knock-out European call option. This type of option has two constant linear barriers designated as the upper and lower barrier. We compare the convergence rate of our method with that of Ritchken's revised trinomial tree method. The results are presented in Table 6. The option value approximations are in good agreement. While Ritchken's method converges to approximately 1.46, the probability adjustment method converges towards 1.47. The difference is slight and can be attributed to the inherent difference in the convergence properties between the (node-repositioned) trinomial and binomial lattice methods. Specifically, we are not pricing with the barrier coincident on the lattice nodes whereas Ritchken is. Thus, while fluctuations about the analytical value are quickly eliminated using a trinomial lattice, these oscillations are damped out at a much slower rate when using a binomial lattice. Nonetheless, the probability adjustment technique is easily implemented and can be readily used to price a double knock-out European call option to within 1% of the analytical price after 5000 time divisions.

Number of Time Steps in the Tree	**Double Knock-Out Call Value (Ritchken, 1995)**	**Double Knock-Out Call Value (Probability Adjustment)**
50	1.418	1.568
75	1.427	1.537
100	1.441	1.552
150	1.446	1.521
200	1.450	1.475
400	1.454	1.474
800	1.456	1.472
1000	1.456	1.479
2000	1.457	1.488
3000	1.457	1.467
4000	1.458	1.474
5000	1.458	1.472

Table 6: Comparison of the convergence results for a double knock-out European call option. Parameters for the option are $S_0 = 95$, $K = 100$, $\sigma = 25\%$, $r = 10\%$, $T = 1$ year, upper barrier = 140, and lower barrier = 90.

We next analysed our method against the option price values calculated using the inverse Laplace transform technique of Geman and Yor (1996). We calculated our option price approximations for the three scenarios provided in their paper. The results of the calculations from 1000 to 5000 time divisions are compared to the analytical values and summarized in Table 7. The

Number of Steps in Tree	**Option Value Approx. Case 1**	**Option Value Approx. Case 2**	**Option Value Approx. Case 3**
1000	0.0414	0.0184	0.0774
2000	0.0412	0.0181	0.0765
3000	0.0412	0.0182	0.0767
4000	0.0413	0.0181	0.0765
5000	0.0411	0.0181	0.0765
Analytical Value (Geman and Yor, 1996)	0.0411	0.0178	0.07615

Table 7: Comparison of the probability adjusted option value approximations to the analytical values calculated using the inverse Laplace transform technique of Geman and Yor (1996). Case 1 parameters are: $S_0 = 2$, $K = 2$, $\sigma = 20\%$, $r = 2\%$, $T = 1$ year, lower barrier = 1.5, upper barrier = 2.5. Case 2 parameters are: $S_0 = 2$, $K = 2$, $\sigma = 50\%$, $r = 5\%$, $T = 1$ year, lower barrier = 1.5, upper barrier = 3. Case 3 parameters are: $S_0 = 2$, $K = 1.75$, $\sigma = 50\%$, $r = 5\%$, $T = 1$ year, lower barrier = 1, upper barrier = 3.

difference between the option value approximation (for 2000 and greater iterations) and the value computed by Geman and Yor is small. In most cases we observe agreement to 3 decimal places. This pattern is consistent for a range of volatilities, strikes, interest rates and barrier levels as evidenced by the three distinct parameter scenarios. We can conclude that the performance of the method lies somewhere between the "stretched" trinomial tree technique of Ritchken (1995) and the "revised binomial" tree method of Boyle and Lau (1994). The advantage lies in the relatively simple implementation of the adjusted probability tree as it closely parallels the construction of a standard binomial tree. Furthermore, convergence rates are improved compared to the CRR model and differences between the option price approximation and the analytical value are in the range of 1 or 2 percent.

As a final example with constant barriers, we consider the convergence curve of a double knock-out European call option. We calculate the option price approximation for all time divisions, inclusive, from 1 to 1000. Along with the price convergence, we also include on the graph what we term the "barrier distance measure" or *BDM*. This value indicates the distance from the midpoint between two nodes in the same time division to the barrier. Thus, for a given node (i, j), barrier level BL_j and stock price $S_{i,j}$, the barrier distance measure, BDM, is computed as

$$BDM = \frac{1}{2}\frac{\left|\left(BL_j + BL_{j+1}\right) - \left(S_{i,j} + S_{i+1,j+1}\right)\right|}{\left(S_{i,j} - S_{i+1,j+1}\right)} \tag{14}$$

Consequently, the distance measure provides an indication of the quality of the option price approximation. Where the distance measure exhibits a local maximum we have a good approximation to the analytic price whereas a local minimum is a poor approximation since the measure indicates the distance from the midpoint. To illustrate the behaviour of the price approximation in relation to the BDM, consider the diagram pictured in Figure 5. For barrier B', the probability adjustment applies and we can expect an improved price approximation from the adjusted model whereas the standard binomial tree model produces a poor approximation to the option price. For barrier B'', both the adjusted and regular binomial trees produce a good approximation to the option price. Conversely, at the midpoint, M, both models produce poor approximations to the analytic value of the option. However, because of the transition probability adjustment at B', the difference between the adjusted price and the analytic price is smaller than the difference between the unadjusted binomial tree price and the analytic price. Thus, the adjusted model is able to produce consistently better estimates of the option price than the Cox-Ross-Rubinstein model. Clearly, as discussed, lattice techniques are much more accurate when the barrier lies on (or close to) the nodes of the lattice.

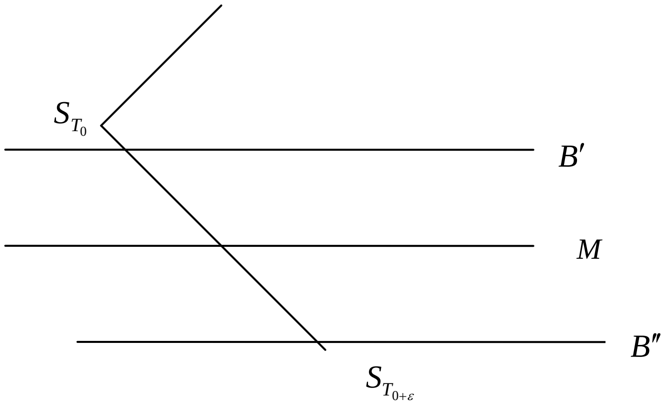

Figure 5: Graphical depiction of the possible barrier positionings with respect to the binomial tree lattice nodes.

The results of our convergence calculation are presented in Figure 6. From the figure, one can easily see the improved convergence rate towards the analytical value exhibited by the probability adjustment method compared the slow convergence rate of the standard binomial tree method. Oscillations of the probability-adjusted approximation begin to be damped out after about 250 time divisions of the tree whereas the oscillations of the regular binomial method continue quite prominently. Furthermore, oscillations within a given crest of the convergence curve are smaller in magnitude that those of the binomial model. It should be noted that in the figure the distance measure is a summation of the individual distances from the midpoint to both the upper and lower barriers. In comparing the crests of the barrier distance measure to the adjusted probability curve, it is evident that the local maximums correspond to the local minimums of the convergence curve. As mentioned previously, these points correspond to the most accurate option price approximations. It is interesting to note the lengthening period of the BDM curve. This provides a graphical illustration of the convergence behaviour of empirical pricing methods based on the CRR binomial tree model.

As one further example, we consider a double knock-out option with short-term maturity. The results are presented in Table 8. It is clear from the table that for short-term maturities (1 month) the probability-adjusted method is in good agreement with the values indicated using the abbreviations by “KI”, “Ana” and “FD”[7]. We conclude that for short-term maturity options, the model demonstrates accurate approximations to the option price within 2000 time-divisions of the lattice. Furthermore, these results do not differ significantly from those calculated using 1000 divisions of the tree.

Vol	U	L	KI	Ana	FD	Approx 1	Approx 2
$\sigma = 0.2$	1500	500	25.12	25.12	24.57	25.12	25.12
	1200	800	24.76	24.76	24.69	24.77	24.76
	1050	950	2.15	2.15	2.15	2.18	2.17
$\sigma = 0.3$	1500	500	36.58	36.58	36.04	36.59	36.58
	1200	800	29.45	29.45	29.40	29.52	29.46
	1050	950	0.27	0.27	0.27	0.28	0.28
$\sigma = 0.4$	1500	500	47.85	47.85	47.31	47.86	47.85
	1200	800	25.84	25.84	25.82	25.89	25.94
	1050	950	0.02	0.02	0.01	0.02	0.02

Table 8: Comparison of the adjusted method with values from Pelsser (1997). The option parameters used are S_0 = 1000, K = 1000, r = 5% and T = 1/12. Note: “Approx 1” is calculated for 1000 time-divisions of the adjusted tree and “Approx 2” is calculated using 2000 time-divisions of the adjusted tree. Column “U” gives the upper barrier while column “L” gives the lower barrier. KI is calculated using the method of Kunitomo and Ikeda (1992), “Ana” are the results of Pelsser (1997) and “FD” is the finite difference calculation based on a 1000 by 1000 grid.

In the next section we will consider the results of the valuation of barrier options with time-varying barriers.

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[7] See Table 8 caption for an explanation of the abbreviations used.

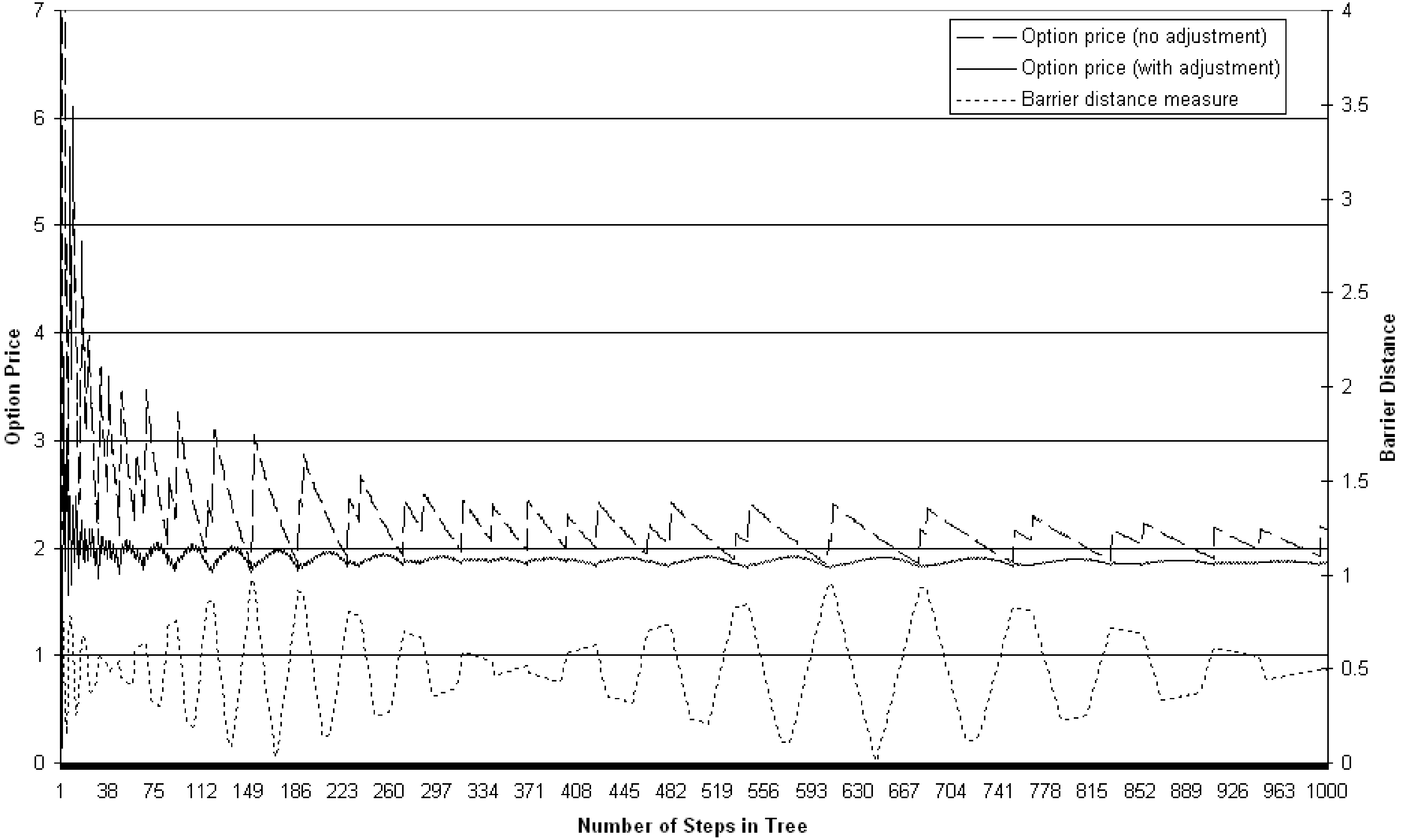

Figure 6: Results of the convergence calculation for a double knock-out European call option. Parameters for the calculation are $S_0 = 100$, $K = 80$, $\sigma = 25\%$, $r = 10\%$, $T = 1$ year, upper barrier = 120, lower barrier = 85.

3.3 Time-varying Barrier Options

3.3.1 Exponential Barrier Option

We now turn our attention to the case of a barrier option with a time-varying exponential barrier. The barrier is characterized by its slope and intercept. Having specified the barrier and the parameters of the option we calculated the option price approximation for all time divisions, inclusive, from 1 to 1000 and plotted the results in Figure 7.

After approximately 600 time-divisions of the tree, we note the decreased oscillations of the probability-adjusted model compared to that of the Cox-Ross-Rubinstein model. Convergence towards the analytical value of 5.4861 is also faster. Due to the nature of the non-linear barrier, the barrier distance measure does not aid in identifying the quality of the approximation and has been left-out of the plot. To judge the performance of our model for the exponential barrier option, we compare to the empirical results published by Costabile (2002). The results are shown in Table 9.

	Slope = δ = -0.1			**Slope = δ = 0.1**	
Tree Lvls.	**Costabile**	**Adjusted**	**Tree Lvls.**	**Costabile**	**Adjusted**
17	7.002	6.841	**24**	5.020	5.227
77	6.958	6.871	**92**	4.949	5.091
181	6.920	6.920	**203**	4.934	5.041
327	6.910	6.930	**356**	4.934	5.016
515	6.912	6.935	**552**	4.932	4.999
2100	6.902	6.927	**2174**	4.929	4.964
4754	6.900	6.919	**4865**	4.929	4.952
Analytic	**6.896**		**Analytic**	**4.928**	

Table 9: Comparison of results between the adjusted-probability method and the extended Cox-Ross-Rubinstein method of Costabile (2002).

In the $\delta = 0.1$ case, Costabile's method converges faster than the adjusted probability technique due to the fact that he is repositioning the nodes of his lattice where we are not. As mentioned, we know that methods based on lattice manipulation exhibit faster convergence. Nonetheless, Costabile's method is specific to exponential barrier options only but the adjusted method can handle a multitude of barrier types. In the next section we consider an option with a single linear time-varying barrier.

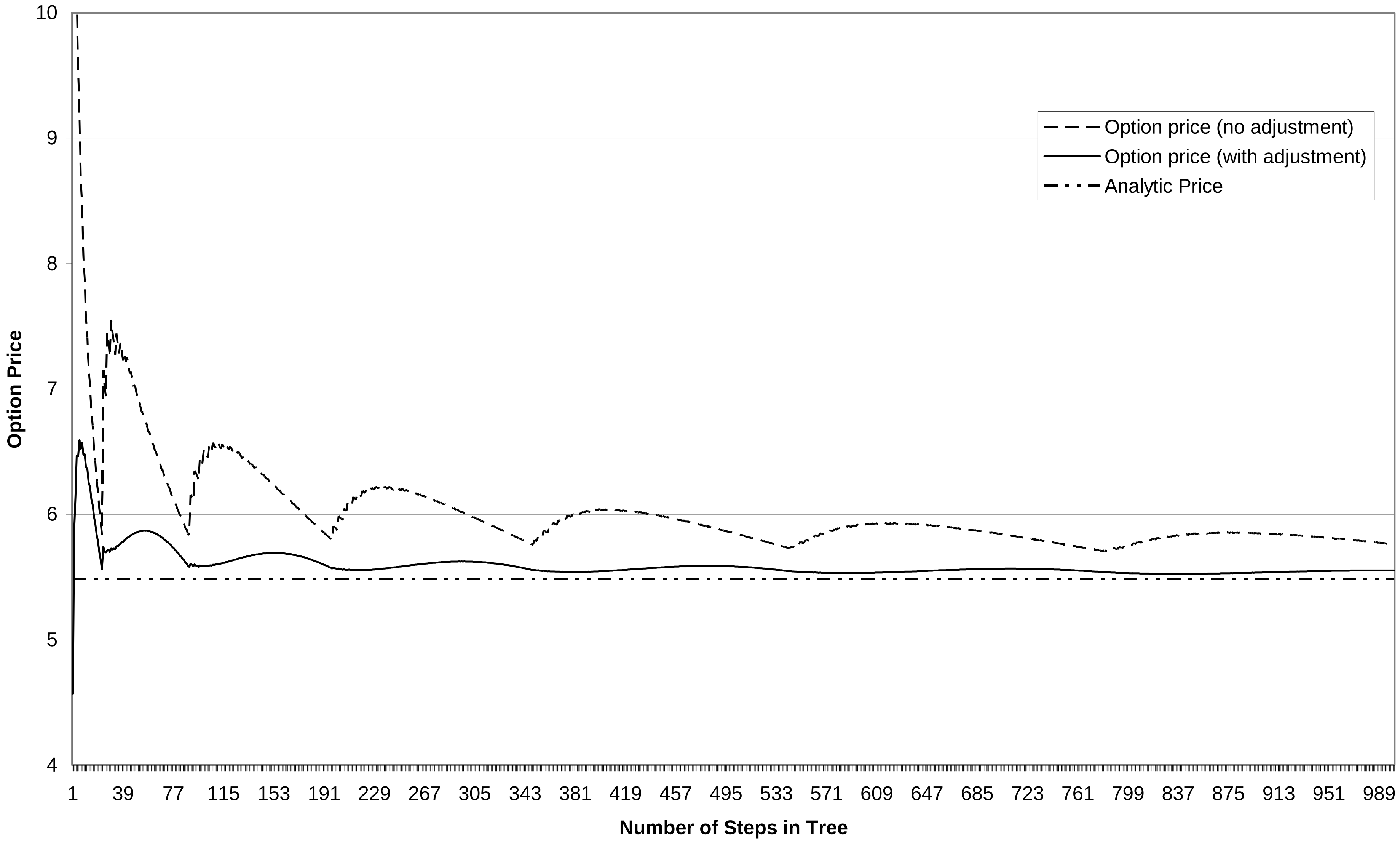

Figure 7: Results of the convergence calculation for an exponential barrier option. Parameters for the calculation are S_0 = 95, K = 100, σ = 25%, r = 10%, T = 1 year, exponential barrier slope = 0.05, exponential barrier intercept = 4.4998. The analytical value is 5.4861.

3.3.2 Single Time-Varying Linear Barrier Option

We now define the barrier of the option as a linear equation. This corresponds to the situation of a single linear (time-dependent) barrier. The barrier equation is specified as

$$L = l_1 t + l_0 \qquad (15)$$

where l_1 is the slope and l_0 is the intercept. In Figure 8, we plot the convergence curve for the linear barrier option with parameters $S_0 = 100$, $K = 100$, $\sigma = 25\%$, $r = 10\%$, $T = 1$, $l_1 = 10$ and $l_0 = 95$. It is immediately clear that the convergence of the adjusted tree is much faster than that of the unadjusted tree. Furthermore, the small fluctuations visible in the crests of the unadjusted curve are almost non-existent in the adjusted price curve. In this case, the BDM assists in identifying those approximations which occurs when the average barrier-node distance is minimal. Unfortunately, for this case we do not possess an analytical solution to which we can compare our results. Consequently, we consider the final case by including an additional linear barrier.

3.3.3 Double Time-Varying Linear Barrier Option

Extending the previous case, we add an additional barrier to the option having the same specification as given in equation (15). We select the parameters to be $S_0 = 100$, $K = 100$, $\sigma = 25\%$, $r = 10\%$, $T = 1$, $l_1 = -22$, $l_0 = 92$, $u_1 = 35$ and $u_0 = 105$. Convergence curves for the double linear time-varying barrier are plotted in Figure 9. We do not have an analytical value for the double time-varying barrier option, yet it is clear that the plot converges to some constant value. Notably, the unadjusted curve converges much more slowly to, presumably, the same value. However, an analytical or independent empirical result is needed to confirm the accuracy of our calculation. As of now, we are unaware of any such work.

Conclusion

In this paper we have presented a very general method for pricing numerous types of barrier options that is based on a modification of the Cox-Ross-Rubinstein binomial tree model. Results for constant barrier options are promising and demonstrate good convergence properties towards the analytical prices. In this way we have shown that it is possible to produce option price

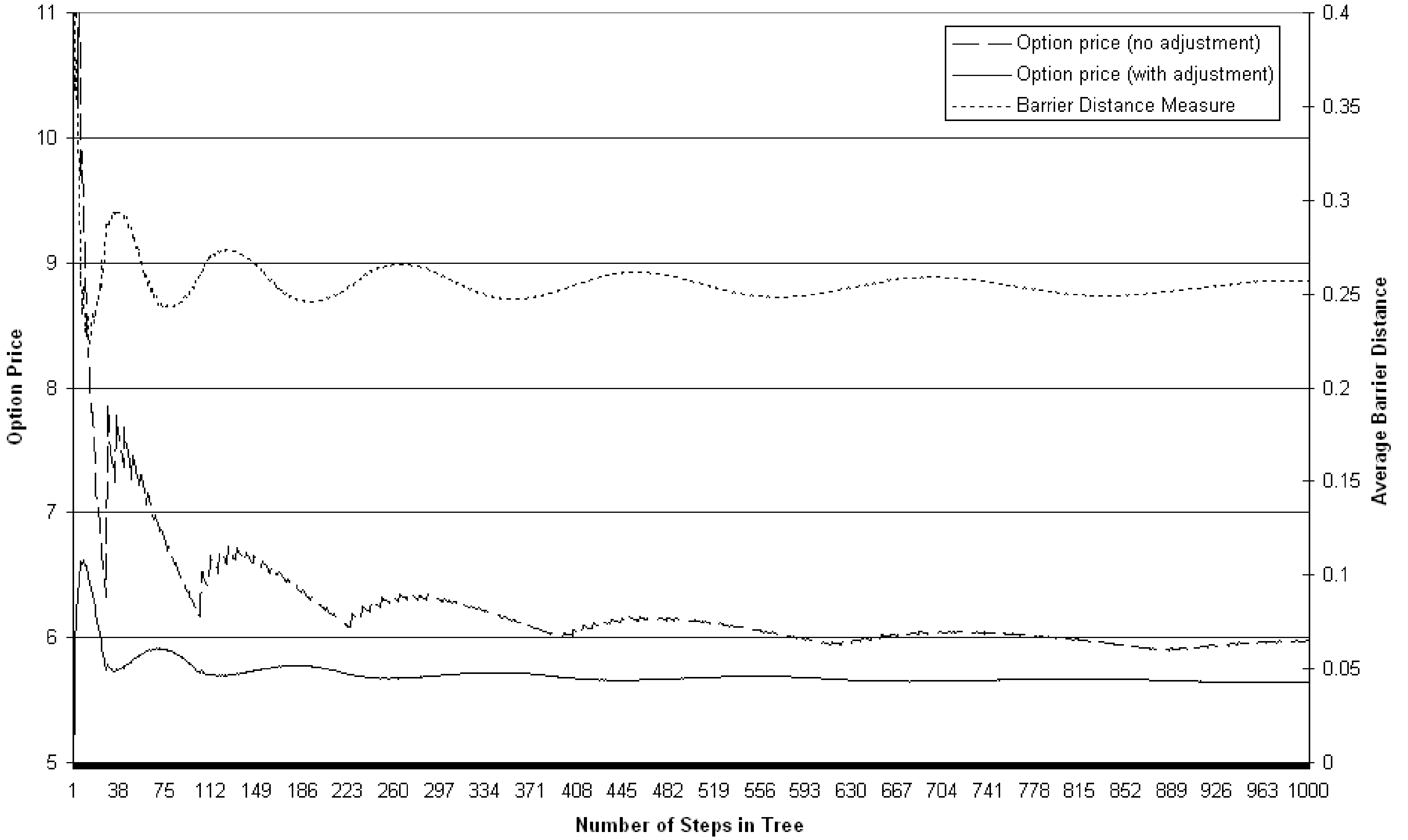

Figure 8: Convergence curves for the case of a single linear time-varying barrier. Option parameters are S_0 = 100, K = 100, σ= 25%, r = 10%, l_1 = 10 and l_0 = 95.

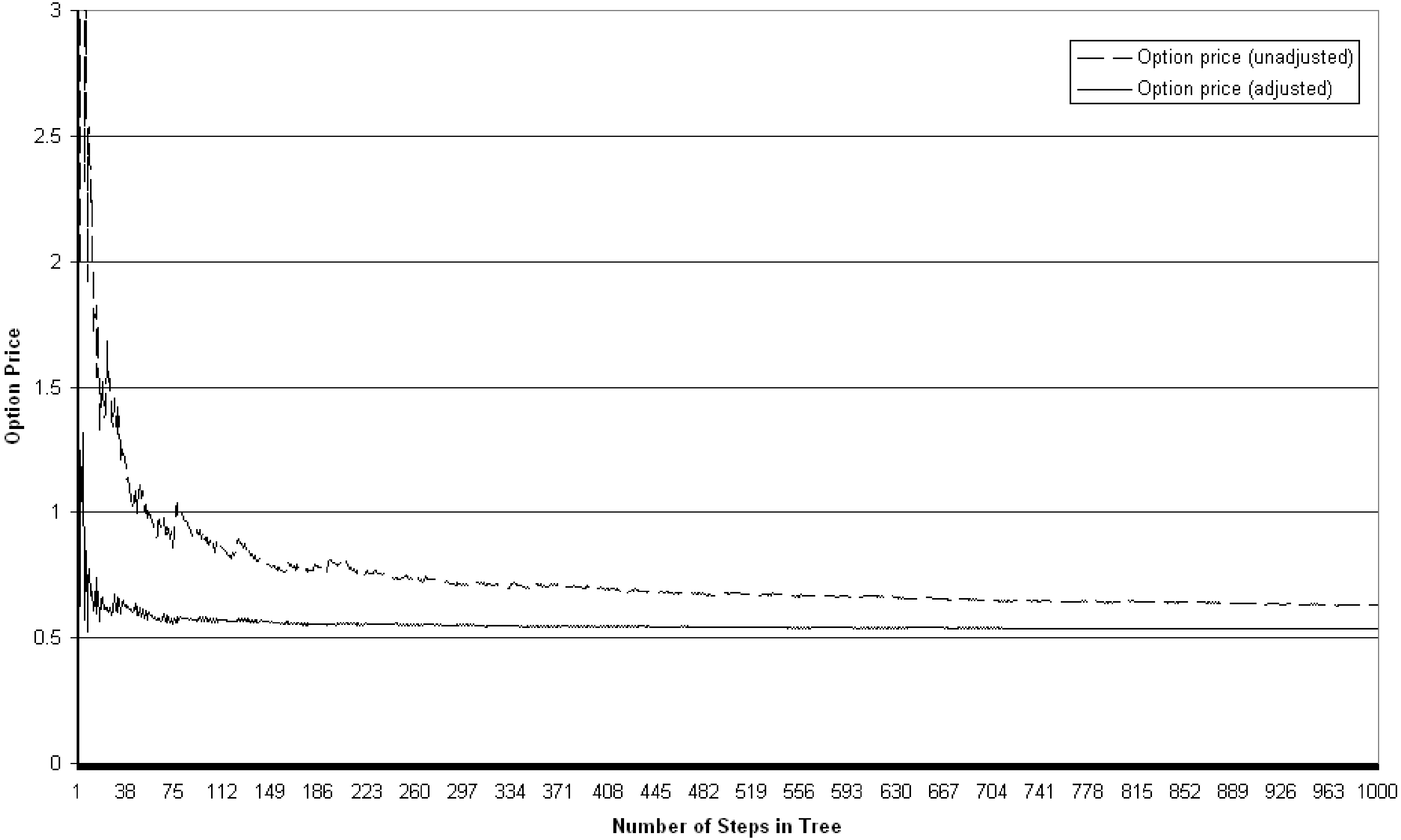

Figure 9: Convergence curves for the double linear time-varying barrier option. Option parameters are $S_0 = 100$, $K = 100$, $\sigma = 25\%$, $r = 10\%$, $l_1 = -22$, $l_0 = 92$, $u_1 = 35$ and $u_2 = 105$.

approximations even when the stock price is very close to the barrier, a result that is difficult to obtain with existing models in the literature. Furthermore, the model produces accurate price approximations for options with short-term maturity.

Furthermore, we have been able to easily extend the method to time-varying barrier options including exponential, single linear and double linear barrier types. A lack of closed-form option valuation equations makes it difficult to gauge the accuracy of the approximation in the cases of the single and double linear time-varying barrier, but nonetheless the price appears to converge to a fixed value. The accuracy of the option price approximations produced by the model dominates that of the Cox-Ross-Rubinstein model while avoiding any repositioning of the lattice nodes. This makes it expedient and simple to implement even for complex option types.

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