AN EMPIRICAL COMPARISON OF CONVERTIBLE BOND VALUATION MODELS

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Abstract

This paper empirically compares three convertible bond valuation models. We use an innovative approach where all model parameters are estimated by the Marquardt algorithm using a subsample of convertible bond prices. The model parameters are then used for out-of-sample forecasts of convertible bond prices. The mean absolute deviation, which is calculated as the absolute difference between the model and the market price expressed as a percentage of the market price, is 1.70% for the Ayache-Forsyth-Vetzal (2003) model, 1.74% for the Tsiveriotis-Fernandes (1998) model, and 2.12% for the Brennan-Schwartz (1980) model. For this and other measures of fit Ayache-Forsyth-Vetzal and Tsiveriotis-Fernandes models outperform Brennan-Schwartz model.

I. Introduction

Exchange-listed companies frequently attract capital by issuing convertible bonds. During the period from 1990 to 2003 globally there were more than seven-thousand issues of convertible bonds.[1] An important problem with convertible bonds is that they are difficult to value. This is caused by the fact that the exercise of the conversion right requires the bond to be redeemed in order to acquire the shares. For this reason a conversion right is in fact a call option with a stochastic exercise price. In addition most convertible bonds are callable in practice. This means that the issuing firm has the right to pay a specific amount, the call price, to redeem the bond before the maturity date. In some convertible bond contracts the *call notice* period is specified thus requiring firm to announce the calling date well before the redemption can be performed. Often times, the call notice period is combined with a *soft call* feature where bond can only be called if the underlying stock price stays above some level for specified period. The valuation process for convertibles becomes substantially complicated as more and more options are accounted for. Thus, it is only possible to incorporate some of many contractual arrangements in the valuation procedure without complicating the latter too much.

Despite the importance of convertible bond valuation for both academic and practical purposes, there is not much empirical literature on this topic. This paper aims to fill this gap by empirically comparing three different convertible bond valuation models for a large sample of Canadian convertible bonds.

Convertible bonds are issued by corporate issuers and thus are subject to the possibility of default. There are two main approaches for valuing securities with default risk. The first approach, called *structural approach*, assumes that default is an endogenous event and

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[1] See Loncarski, ter Horst and Veld (2006).

bankruptcy happens when the value of firm's assets reaches some low threshold level. This approach was pioneered by Merton (1974) who assumed that the firm value followed a stochastic diffusion process and default happened as soon as the firm value fell below the face value of the debt. However, as pointed by Longstaff and Schwartz (1995), a default usually happens well before the firm depletes all of its assets. Valuation of the multiple debt issues in Merton (1974) is subject to strict absolute priority where any senior debt has to be valued before any subordinated debt is considered. This creates additional computational difficulties for valuing defaultable debt of firms with multiple debt issues. Moreover, the credit spreads implied by the approach of Merton (1974) are much smaller than those observed on financial markets. In contrast, a default in the Longstaff and Schwartz (1995) model happens before the firm exhausts all of its assets and as soon as the firm value reaches some predefined level common for all issues of debt. The values of the credit spreads predicted by their model are comparable to the market observed spreads. The common default threshold for all securities allows valuation of multiple debt issues.

In the structural approach debt is viewed as an option on the value of firm assets, and an option embedded in the convertible bond can be viewed as a compound option on the value of firm assets. Therefore, the Black and Scholes (1973) methodology can be used for valuing convertible bonds. The models of Brennan and Schwartz (1977) and Ingersoll (1977) apply the structural approach for valuation of convertible bonds. In these models the interest rates are assumed to be non-stochastic. Brennan and Schwartz (1980) correct this by incorporating stochastic interest rates. However, they conclude that for a reasonable range of interest rates the errors from the non-stochastic interest rate model are small and for practical purposes it is preferable to use the simpler model with non-stochastic rates.

Nyborg (1996) argues that one of the main problems inherent in implementation of structural form models is that the convertible bond value is assumed to be a function of the firm value, a variable not directly observable. To circumvent this problem, some authors model the price of convertible bonds as a function of the stock price, a variable directly observable in the market. The model of McConnell and Schwartz (1986) is such an example. In this model they price Liquid Yield Option Notes (LYONs), which are zero coupon convertible bonds callable by the issuer and putable by the bondholder. However, one of the main drawbacks of their model is the absence of a bankruptcy feature.

The second approach used for valuing the defaultable corporate obligations is the *reduced-form* approach. In contrast to the structural approach, where default is caused by the firm value diffusing to some low level, in the reduced-form models default is an exogenous event caused by a jump of the stock value to zero and not by a diffusion of the firm value to a certain boundary. At any point in time the probability of default is described as a Poisson arrival process and is characterized by a hazard function. The application of this approach for valuing defaultable non-convertible bonds can be found in the models of Jarrow and Turnbull (1995), Duffie and Singleton (1999), and Madan and Unal (2000)[2]. The attractiveness of this approach is that the convertible bond value can be modeled as a function of the stock price. The models of Tsiveriotis and Fernandes (1998), Takahashi, Kobayashi, and Nakagawa (20012), and Ayache, Forsyth, and Vetzal (2003) use the reduced-form approach for valuing convertible bonds. [3]

[2] See the paper of Andersen and Buffum (2004) for discussion of problems associated with calibration and numerical implementation of reduced form convertible bond valuation models.

[3] Another examples are the models of Cheung and Nelken (1994), Ho and Pfeffer (1996), and Hung and Wang (2002). The Cheung and Nelken (1994) and Ho and Pfeffer (1996) models are two-factor models, where the state

In contrast to the extensive theoretical literature on convertible bond pricing, there is very little empirical literature on this topic. Some researchers use market data to verify the degree of accuracy of their own models. Cheung and Nelken (1994) and Hung and Wang (2002) use market data on single convertible bonds to verify their models. Ho and Pfeffer (1996) use market data on seven convertible bonds to perform a sensitivity analysis of their two-factor multinomial model. King (1986) uses a sample of 103 US convertible bonds and finds that the average predicted prices of the Brennan-Schwartz model with non-stochastic interest rate are not significantly different from the mean market prices. Carayannopoulos (1996) uses the stochastic interest rate variant of the Brennan-Schwartz model. For a sample of 30 US convertible bonds he finds a significant overpricing of the deep-in-the-money convertible bonds. Takahashi, Kobayashi and Nakagawa (2001) use data on four Japanese convertible bonds to compare their model to the models of Tsiveriotis-Fernandes (1998), Cheung and Nelken (1994), and Goldman-Sachs (1994). On the basis of the mean absolute deviation they conclude that their model produces the best predictions of convertible bond prices. Amman, Kind and Wilde (2003) use 18 months of daily French market data and the Tsiveriotis-Fernandes model to find that, on average, market prices of French convertibles are 3% lower than the model-predicted prices.

Most companies that issue convertible bonds don't have ordinary bonds outstanding. For this reason we cannot use ordinary bond parameters such as credit rating when calculating

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variables are the stock price and the credit spread-adjusted interest rate. Cheung and Nelken (1994) assume no correlation between the interest rates and stock price changes; Ho and Pfeffer (1996) assume that the correlation is constant across periods. Hung and Wang (2002) add one more state variable, the default event, in addition to the stock price and interest rate. 90

model prices for convertible bonds. Moreover, other model parameters, such as the underlying state variable volatility, the dividend yield, and the default rate are often not directly observable. Therefore, we use an innovative technique that allows for the calculation of model prices even when the values of the parameters are not observable. We divide our sample into two parts: a historical sample and a forecasting sample. Instead of using the values of the parameters inferred from the plain debt data or underlying stock market data we use the information contained in the historical *convertible* bond prices to estimate the necessary parameters. This approach allows for forecasting the convertible bond prices using the convertible bond price series only. The data from the historical sample are used to calibrate model parameters. We then calculate model prices for the forecasting period and compare these to market prices.

It should be mentioned that estimation procedure becomes very complicated if one wants to account for all features of convertible bond contracts. Lau and Kwok (2004) show that the dimension of valuation procedure increases rapidly if one decides to accommodate the soft-call feature. They also show that the calling period is essentially increasing the optimal call price when the issuers should start calling the bond. This effective call price can be viewed as the original call price multiplied by some call price adjustment, $(1+\pi)$. In our study we only account for call and call notice feature and do not price the soft-call feature. Since our sample does not contain putable bonds, we ignore this feature too. We model the call notice period feature by including and calibrating the excess calling cost parameter.

First, we estimate the Brennan-Schwartz model. However, since it is a structural form model, it requires simultaneous estimation of all other defaultable assets. This fact complicates estimation of the model. To eliminate this complication, we use a subsample of firms with a simple capital structure for the estimation of the Brennan-Schwartz model. A simple capital structure is defined as a capital structure that only consists of equity, risk-less straight debt, and convertible debt. The assumption of a simple capital structure substantially simplifies the estimation process. However, it also reduces the domain of applicability for this model.

In order to be able to value the convertible bonds of firms with a non-simple capital structure we need to rely on the reduced-form approach as it is not dependent on the capital structure of a firm. For this reason we use two other convertible bond valuation models. The first is the Tsiveriotis-Fernandes model, from now on to be referred to as the TF-model. The second is the model of Ayache, Forsyth, and Vetzal (2003), from now to be referred to as the AFV-model.

We find that the mean absolute deviation of the model price from the market price, expressed as a percentage of the market price, is the smallest for the AFV-model (1.72.29%). This deviation is 1.743.08% for the TFBS-model, and 2.124.08% for the BSAFV-model. The BSTF-model also shows the smallest range of pricing errors. For the TF and AFVall models we find a negative relationship between moneyness and both absolute and actual values of the pricing errors while this relationship is positive for the BS-model. This means that the AFV and TF-models misprice convertibles that have in-the-money conversion options less than convertibles with conversion options that are at-the-money or out-of-the-money. We also find the negative relationship between the moneyness and the actual value of the error terms for the BS-model meaning that the model tends to overprice the bonds that are deeper in-the-money. We find a positive relation for the reduced-form models between absolute values of pricing errors and the volatility of the returns of the underlying stocks. The effect of volatility on the absolute deviations in the BS-model is statistically insignificantgives the opposite effect.

The remainder of this paper is structured as follows. In Section II we present different convertible bond valuation models. Section III includes the data description. Section IV is devoted to the estimation of the parameters. The results of the estimation are presented in Section V. The paper concludes with Section VI where the summary and conclusions are presented.

II. Convertible bond valuation models

A. Model selection

A comparison of valuation models is possible if all the model input variables are either directly observable or can be estimated. Structural models that use non-directly observable variables, such as firm value, are very difficult to estimate. Their estimation becomes easier if a simple capital structure of the firm is assumed. Reduced-form models, on the other hand, use directly observable market variables and are much easier to estimate. This reason explains their popularity among practitioners. The selection of models that are used for the comparison in our study is based on popularity with practitioners as well as their sound theoretical underpinnings. In this paper we compare the models of Brennan and Schwartz (BS) (1980), Tsiveriotis and Fernandes (TF) (1998)[4], and Ayache, Forsyth, and Vetzal (AFV) (2003).

B. Convertible bond valuation models

[4] In their overview of convertible bond valuation models, Grimwood and Hodges (2002) argue that the approach of Tsiveriotis-Fernandes is the most popular among practitioners.

The BS-model

Brennan and Schwartz (1977, 1980) develop a structural type approach for valuing convertibles where the convertible bond value is modelled in terms of the firm value. The main assumptions of their approach are: (a) the firm value W is the central state variable, the risk-adjusted return on which is the risk-free rate at each instant; (b) the dilution effect resulting from conversion must be handled consistently; (c) the effect of all cash flows on the evolution of the firm value must be accounted for; (d) assets must be sufficient to fund all assumed recoveries in default; and (e) the share price process is endogenously determined by all this.

The firm value W is assumed to be governed by the stochastic process $\mathrm{dW} = (\mathrm{rW} - \mathrm{D(W)} - \mathrm{rB_s} - \mathrm{cB_c})\mathrm{dt} + \delta \mathrm{Wdw}$ in which r is the instantaneous riskfree interest rate, B_s is the par value of senior straight bonds outstanding, B_c is the par value of convertible subordinated bonds outstanding, c is the annualized continuous coupon rate on convertible bonds, $\mathrm{D(W)} = \partial \max\{0, \mathrm{W} - \mathrm{B_s} - \mathrm{B_c}\}$ is the total continuous dividend payout on shares, δ ∂ is the constant dividend rate on book value of equity, and $\sigma \delta$ is the constant proportional volatility of the asset value. The stochastic process above is applied when the firm is not in default. Following Brennan and Schwartz (1980)[5] we further assume a constant default boundary prior to convertible debt maturity at the firm asset level $\underline{\mathrm{W}} \equiv \mathrm{B_s} + \rho \mathrm{B_c}$, where ρ denotes the convertible bond early recovery rate as a fraction of par. This early default

[5] The original Brennan-Schwartz (1977) paper does not permit early default, while the 1980 paper has an early default boundary.

boundary implies *W* is just sufficient to fund full recovery on the senior straight debt, recovery on the convertibles, and zero recovery on equity at the time of default.[6]

The assumptions described earlier imply, due to the standard arbitrage arguments that the value of the entire convertible bond issue, *V*, has to follow the partial differential equation (PDE) $\frac{1}{2}V_{WW}\delta^2 W^2 + (rW - D(W) - rB_s - cB_c)V_W + cB_c + V_t - rV = 0$ where the subscripts indicate partial differentiation.

Boundary conditions characterize the convertible bond value at maturity, at early default point $\underline{W}$, at rational early conversion level W^*, and at rational early call level of *W* if applicable. These conditions are as follows:

$$V(W,T) = \begin{cases} \max\{B_c, C(W)\} \text{ for } W \geq B_c + B_s \\ \max\{0, W\text{-}B_s\} \text{ for } W < B_c + B_s \end{cases} \quad \text{(maturity)}$$

$$V(W,t) \geq C(W) \text{ for all W, t} \quad \text{(voluntary conversion)}$$

$$V(\underline{W},t) = \rho B_c \quad \text{(early default)}$$

$$V(W,t) = \max\{P_c C(W)\} \text{ for } V \geq (1+\pi)P_c \text{ for all } t \geq T_c \quad \text{(early call)}$$

In the above, *T* is the maturity date of the convertible bond, P_c is the early call price of the bond, T_c is the first call date of the bond, *C(W)* is the conversion value of bond given *W*, and *x*

[6] Note that we treat coupons as being continuously paid for purposes of the evolution of *W*. If the coupons are periodic, this is equivalent to saying that the accruing interest is continuously and irrevocably paid into a segregated escrow account which pays full accrued interest to bondholders in the event of default. In the actual computation, coupons are paid at discrete intervals and (linearly) accrued interest is assumed to be paid to the bondholder at the instant of conversion, call, or default.

is the excess value required for an early call.[7] The excess value π required for early call is introduced to accomodate the call notice period presence and also a frequent empirical observation that firms appear to delay calling bonds until well beyond the fully rational level .[8] Note that upon notice of early call, bondholders exercise the conversion right if that gives a higher value. The BS-model is the only model that allows for the possibility of share dilution after the conversion of convertible bonds. The other models in our study ignore this feature by assuming no dilution after conversion. Solving the above PDE subject to the boundary conditions gives the theoretical value of the entire outstanding convertible bond issue.

There is little guidance on the empirical implementation of the original Brennan-Schwartz (1977, 1980) models. In our study we have filled in the missing elements of the BS-model as simple as possible by: (a) postulating fixed but unobservable senior claims B_s (bonds, bank loans, amounts due to government and suppliers, etc.); (b) specifying dividend flows in a way that they are non-negative, yet embody likely covenants in senior and subordinated debt; (c) selecting early default barrier consistent with assumed risk-free nature of senior debt and assumed partial recovery rate on subordinated debt; (d) assuming floating coupon rate equal to r on senior debt so that its market value is constant over time; and (d) assuming agents expect the risk-free rate $r(t)$ will follow a deterministic path implied by the term structure of Treasury rates at each time t. Note that the share price process implicit in all this cannot exhibit constant

[7] The bond conversion value *C(W)* is defined as follows. Let *K* denote the exercise share price for the convertible bond, and N_0 the number of shares outstanding prior to exercise, and *z* as the fraction of firm assets, net of senior debt, owned by bondholders after conversion. $q \equiv \frac{B_c / K}{N_0}$ $z \equiv \frac{q}{1+q}$. Then, $C(W) = (W - B_s)z$

[8] This might reflect either inattention on the part of management, but could also reflect issue costs associated with the replacement debt or credit market access concerns.

proportional volatility as typically assumed in competing models. Therefore, the stock price volatility will increase as assets fall closer to the default point. Similarly, the proportional dividend yield on the shares varies with the level of assets.

As mentioned before, determining the convertible bond value is a problematic task if the firm has a complex capital structure. The value of the convertible bonds has to be determined simultaneously with the values of all senior claims. In our study, we only estimate values for the BS-model for companies that have a simple capital structure. This is defined as a capital structure that only consists of equity, straight debt, and convertible debt. This means that we have to exclude companies that have preferred equity, warrants, and/or different types of subordinated debt in their capital structure. In this approach, the straight debt is assumed to be riskless. The value of the firm is simply a sum of the values of the equity, convertible debt, and straight debt. This assumption eliminates the necessity of simultaneous valuation of convertibles and senior claims.

The list of observable constant parameters for the model is: c, T, T_c, P_c, K, and N_0. We further observe at each time t the then current risk-free forward rate structure $r(\tau)$, $\tau \geq t$, and the combined market value of shares plus convertibles (identical to firm assets net of senior claims) $W(t)-B_s$. The list of unobserved constant parameters, to be either specified or estimated, is: σ, ρ, $\delta\partial$, π, and B_s. These parameter estimates are chosen via an extended Marquardt algorithm to minimize the sum-squared deviations of theoretical quotes from market quotes for the convertibles. This estimation procedure is, in effect, non-linear least squares, since the predicted quotes are non-linear functions of the parameters being estimated.

The TF-model

The TF-model, which is based on the methodology of Jarrow and Turnbull (1995), discriminates between two parts of the convertible bond: the bond-like or cash-only part (COCB) and the equity-like part. The COCB is entitled to all cash payments and no equity flows that an optimally behaving owner of a convertible bond would receive. Therefore, the value of the convertible bond, denoted as *V*, is the sum of the COCB value, denoted as Σ, and the equity value, (V-Σ). The stock price is assumed to follow the continuous time process $dS = rSdt + \delta S dw$, where *r* is the risk free interest rate, δ is the standard deviation of stock returns, and *w* is a Wiener process. Since the bond-like part is subject to default, the authors propose to discount it at a risky rate. The equity-like part is default-free and is discounted at the risk-free rate. Convertible bond valuation then becomes a system of two coupled PDEs:

For V: $$\frac{1}{2} V_{ss} \delta^2 V^2 + r\, SV_s + V_t - r(V - \Sigma) - (r + r_c)\Sigma + f(t) = 0$$

For Σ: $$\frac{1}{2} \Sigma_{ss} \delta^2 V^2 + r\, S\Sigma_s + \Sigma_t - (r + r_c)\Sigma + f(t) = 0$$

S is the the underlying stock price, r_c is the credit spread reflecting the pay-off default risk; *f(t)* specifies different external cash flows for cash and equity (e.g. coupons or dividends).

In order to find the value of the convertible bond, it is necessary to solve the system of PDEs. At each point in time, the convertible bond prices should satisfy boundary conditions. At the maturity date the following conditions should hold: $\mathrm{V(S,T)} = \max(\mathrm{aS}, \mathrm{F} + \mathrm{Coupon})$, $\Sigma(\mathrm{S,T}) = \max(\mathrm{F}, 0)$ where *a* is the conversion ratio, and *F* is the face value of the bond. At the conversion points the constraints are: $\mathrm{V(S, t)} \geq \mathrm{aS}$; $\Sigma = 0$ if $\mathrm{V(S, t)} \leq \mathrm{aS}$. Callability constraints: $\mathrm{V} \le \max$ (Call Price, aS); $\Sigma = 0$ if $V \ge$ Call Price. Putability constraints are: $V \ge$ Put Price; $\Sigma =$ Put Price if $V \le$ Put Price.

The prices of the convertible bond are first calculated for different stock prices at the maturity date. In the “equity-like” region of underlying stock prices, where the value of the bond if converted is higher than the face value plus accrued coupons, the convertible bond price is equal to the conversion value. In this range the price of the convertible bond is discounted one period back at the risk-free rate, *r*. In the stock price range, where the total of face value and accrued coupon is higher than the conversion value, the convertible bond prices are discounted at the risky rate, r+r_c. Working one period back, the convertible prices are calculated and the points are found where the issuer can call the bond and the holder can put the bond. The iterations continue until the initial date is reached.

The AFV-model

The AFV-model is a modified reduced-form model that assumes a Poisson default process. The authors of this model argue that the TF-model does not properly treat stock prices at default, because it does not stipulate what happens to the stock price of a distressed firm in the case of bankruptcy.

The AFV-model boils down to solving the following equation $MV - p\max(aS(1-\eta), RX) = 0$, where MV is defined as following

$MV \equiv -\frac{1}{2}V_{ss}\delta^2 V^2 - (r + p\eta - d)SV_s - V_t + (r + p)V$, subject to the boundary conditions

$\mathrm{V} \le \max$ (Call Price, aS), $V \ge$ Put Price, where p is the probability of default, η is the proportional fall in the underlying stock value after a default occurs.

In the original paper (Ayache et al., 2003) authors argue that X can take many forms, be it face value of bond, an accreted value of the zero coupon bond or the pre-default market value of the bond. In our study we use one of the versions of the AFV-model where we assume X to be equal to the bond's face value. Thus, *R* is the proportion of the bond face value that is recovered immediately after a default, and *d* is the continuous dividend rate on the underlying stock.

The AFV-model assumes the probability of default to be a decreasing function of stock price: $p(S) = p_0^{'}(\frac{S}{S_0})^{\alpha}$ The symbols S_0, $p_0^{'}$ and α represent constants for a given firm; $p_0^{'}$ is the probability of default when the stock price is S_0. We can also group S_0 and $p_0^{'}$ together, introduce $\gamma = \frac{p_0^{'}}{S_0^{\alpha}}$, and therefore rewrite the nazard function as $p(S) = \gamma S^{\alpha}$

III. Data description

In our study, we use the sample of 97 actively traded Canadian convertible and exchangeable bonds listed on the Toronto Stock Exchange as of November, 1 2005.[9] None of the bonds in the sample is putable. These bonds have different issue dates going back from October 2005 to April 1997. The maturity dates range from March 2007 to October 2015. Fifty

[9] The pricing data for the convertible bonds in our study comes from the Toronto Stock Exchange. All historical price series reflect the actual prices and are not derived by extrapolation. This allows us to avoid the common problems associated with the pricing of privately traded bonds.

six bonds were issued by income trusts.[10] This number is surprising since convertible bonds are tend to be issued by young and growing firms while income trust structure is more suited to stable, mature firms (Halpern, 2004). Forty-three convertible bonds of 56 were issued by income trusts operating in oil and gas and real estate industries. The detailed characteristics of the bonds used in the study can be viewed in Appendix A.

Even though there should be no difference in pricing of convertible bonds issued by income trusts and ordinary corporations we will briefly outline the essence of income trusts as there are no corresponding securities in US market. Income trusts can be viewed as subsample of mutual funds. They raise funds by issuing units of securities to public and purchase most of the equity and debt of successful businesses with acquired funds. Thus, operating businesses act as subsidiaries of income trusts which in turn distribute 70% to 95% of cash flows to unit holders as cash distributions (Department of Finance, Canada, 2005). Since most of the earnings are distributed to unit holders, little or no funds are left for research and development and capital expenditures the stable and mature types of businesses deemed most suitable for an income trust structure. Preferential tax treatment of publicly traded investment vehicles makes income trusts a widespread business structure in Canada.

For the purpose of finding the convertible bond valuation model which predicts the prices that are the closest to market prices, the data is divided into two sub-samples: historical and forecasting. The pricing data that we use is weekly data with prices being observed every Wednesday (to ensure high activity on the market). The models' parameters are calibrated using the data from the historical sub-sample. Then, weekly model prices are calculated for

[10] An income trust is an investment trust that holds assets which are income producing. The income earned on the assets is passed on to the unit holders.

each convertible bond for the forecasting period using calibrated parameters. The best model is selected based on the distance between the actual forecasting period market prices and the model predicted prices for the forecast period.

The historical sub-sample for each bond starts with its issue date and ends at the start of the forecast period. The beginning date of the forecast period for each bond varies with the bond's issue date. In our study, we define that the historical sub-period in a way so that it is no shorter than one year. If the bond was issued before January 1, 2004, then the forecast period begins on January, 1 2005. If the bond was issued between January 1, 2004 and July 1, 2004 the starting date is set to July 1, 2005. If the bond was issued after July 1, 2004, the starting date of the forecast period is set to January 1, 2006. Thus, the bonds with earlier issue dates have longer forecast sub-samples. This choice of starting date is stipulated by the need of a large enough historical sub-sample for the estimation of the model parameters.[11] TWe fix the end of the forecasting period is fixed on April 28, 2006. We equally weight the errors from observations in historical sub-sample as the approach we use assumes that parameters stay constant over time, thus enabling us to use the calibrated values of parameters for out-of-sample convertible bond price predictions.

From the original sample of 97 bonds, we exclude all exchangeable bonds as well as bonds traded in currencies other than the Canadian dollar. After screening for the issues that

[11] Loncarski, ter Horst and Veld (2007) find that during the first six months after their issue convertible bonds are underpriced. This provides a possibility for convertible arbitrage. To avoid pricing biases we perform an alternative pricing procedure which uses reduced historical samples where the first six months of data are dropped. The results of the reduced sample estimation are similar to the ones using the original data samples. These results are available from authors on request.

have price series and prospectuses available, as well as information on underlying stocks and financial statements with dividend information, the sample reduces to 64 issues. Seventeen firms from this sample have a simple capital structure consisting of equity, straight debt, and convertible debt only. These bonds are used for estimating the BS-model. Detailed information concerning the issues used in the study can be found in Appendix A.[12]

Globe Investor Gold database provides data on historical bond prices. We take detailed information on each issue including coupon rates, maturity dates, and conversion conditions from the prospectuses available at the SEDAR (*System of Electronic Document and Archive Retrieval*) and Bloomberg databases. We use Dominion Bond Rating Service data on existing debt and issuer ratings. The information on underlying stocks' dividends comes from companies' websites and from the Toronto Stock Exchange. The information on the number of stocks and convertible bonds outstanding is taken from the Canadian Financial Markets Research Centre database. The descriptive statistics of the convertible bond characteristics are presented in Table 1I.

[Please insert Table I here]

As can be seen from Table I1, the shortest time to maturity for the bonds in the sample is 1.25 years while the longest time to maturity is almost 10 years; the average time to maturity is around 5 years. The average degree of moneyness of the bonds for the sample period, S/K,

[12] The assumption of a simple capital structure reduces the sample to 17 firms. This subsample is only used only for evaluation of the Brennan-Schwartz model. All other models are evaluated using the complete sample of 664 bonds.

ranges from 0.30 to 2.46. The average bond is slightly in-the-money with a ratio of the underlying stock price to the exercise price of 1.06. The least volatile underling stock has an annual standard deviation of 13%, the most volatile – 60%. The average volatility of the underlying stocks is 25%. The average coupon rate for the convertible bonds in the sample is 7.2%.

Since in the evaluation of the models risk-free interest rates are used, we use the forward interest rates, derived from the Bank of Canada zero coupon bond curves, as a proxy. The forward rates used are 3-month forward rates for horizons from 3 months to 30 years. Zero-coupon bonds data is taken from the Bank of Canada.

In contrast to the estimation techniques that require use of the firms' straight corporate bonds for estimating model parameters, we employ the method that uses information inherent in the *convertible* bond prices for calibrating the parameters of the models. Many of the firms in our sample issue convertible debt instead of ordinary bonds in order to save the costs of interest in the absence of a high credit rating. These young and growing firms offer investors convertible bonds with lower coupons. In exchange for these lower coupons, the conversion feature is added. The majority of these firms do not have other publicly traded corporate bonds in their capital structure. Therefore, using a method for convertible valuation that does not hinge on the presence of the firm's straight corporate debt promises to be valuable.

The information contained in the prices of the convertible bonds may be helpful to calibrate parameters of the models in the absence of other forms of bonds for the firm. Moreover, by using information contained in the historical convertible price series, it is possible to estimate all other convertible bond parameters such as the underlying state variable (stock price or firm value) volatility, dividend yield, and diffusion processes parameters.

Using historical convertible prices, we employ the Marquardt algorithm (Marquardt, 1963) to search for the model parameters that minimize the squared sum of residuals between model predicted prices and market prices. Later, we use these parameters for forecasting the convertible prices for our forecasting sub-sample. Initial values and boundaries for parameters are provided based on the assumption of the corresponding models.

Given the convertible valuation model, the Marquardt algorithm finds the theoretical convertible bond prices given the initial values for model parameters. In the next step, the algorithm changes the model parameters until the values that return the minimum squared deviations of the model prices from the observed market prices are found.[13] The data needed for the estimation of the parameters and prediction of the out-of-sample theoretical convertible bond prices consist of the convertibles' market prices, conversion prices, issue-, settlement- and maturity dates, coupon rates, number of coupons per year, market prices of the underlying stock, call schedules and call prices, and numbers of outstanding convertible bonds and stocks for the BS-model.

We limit the parameters' values to ensure that Marquardt algorithm does not assume unrealistic or non-plausible parameter values. The parameter σ (volatility) has a lower floor of zero and an upper floor of 1 (representing standard deviation of 100% of stock price or firm value); ρ (bond recovery fraction) is assumed to be between 0 and 1; dividend yield (d) is assumed to be between 0 and 30%, call price adjustment parameter π to be between 0 and 30% of the call price, debt value for BS model to be between zero and 10 million dollars, credit spread to have a lower floor of zero. It is impossible to guarantee that the estimated parameters

[13] This technique allows estimation not just of the parameters of the hazard function, but also of the characteristics of the convertible bonds such as volatility and dividend yields that are implied by the convertible prices.

represent the global minimum for the algorithm minimization score without complicating the estimation algorithm too much. To ensure that the calibrated parameters are robust to the starting values we repeat the calibration for each firm several times trying out different starting values until we find parameter set that represent the lowest value of squared residuals between the algorithm predicted prices and the market prices. The descriptive statistics of the model parameters calibrated can be seen in Table II.

[Please insert Table II here].

It can be seen that few of parameters calibrated by the models are in the range consistent with the real market observations. The implied underlying return stock volatility (annualized standard deviation) is 33% for AFV and TF-models and the implied volatility of firm value is 43% for BS-model. The calibrated dividend yield is on average equal to 21% for AFV and TF-models and 26% for the BS-model. These numbers are reasonable considering that the majority of the bonds in the sample are issued by income trusts that historically have high cash distribution yields. The price implied recovery rates are on average 3% for AFV model, 19% for TF-model and 10% for BS-model and are lower than the conventional market values.

Note that, together with convertible bond data, data on ordinary bonds can be used to calibrate the parameters. In this case, the “conversion price” of ordinary debt has to be specified as some unrealistically large number and “call dates” have to be set after the maturity date. This parameter calibration approach may yield superior results compared to the exploitation of only corporate bond data, because the former approach uses a wider set of market information. However, since most of the firms in our sample do not have straight corporate debt, we only use convertible bonds' prices for calibration of parameters in our study.

IV. Estimation

To be able to predict the theoretical convertible bond prices, parameters such as the underlying state variable volatility, dividend yield, and the credit spread are needed. Many of these parameters are not directly observable. Parameters like credit spread may be observed from the straight debt of the same firms. However, as we mentioned before, some firms issue only convertible bonds. For this reason we use an approach where we use historical *convertible* bond prices for estimating *all* necessary parameters of the models.

For the Brennan and Schwartz (1980) model, the dividend payout is assumed to be a fixed proportion of the amount by which the firm value exceeds the principal owed on the debt (this is the sum of straight debt and convertible debt).[14] For the other models, the dividend yield is simply estimated as a constant proportion of the price of the underlying stocks. To account for the call notice period feature we introduce the call price adjustment parameter π; multiplying the original call price specified in bond prospectus by $1+\pi$ gives us the *effective call price* as in Lau and Kwok (2004). This parameter is unknown and is calibrated for all models together with dividend yield (d), stock/firm value volatility (σ), credit spread for TF model (r_c), default bond value recovery fraction (ρ), and hazard process parameters (α, γ) from the historical convertible bond prices. The dividend yield is assumed to be a constant

[14] Note that this is consistent with the numerical illustration of Brennan and Schwartz (1980). In addition, it has an advantage compared to their approach since their dividend specification can lead to negative dividends, while ours does not. Furthermore, it forces dividend payouts to stop while the firm value is still sufficient to repay senior debt completely, justifying our treatment of straight debt as riskless.

proportion of the book value of the shares. The other unobservable variable necessary for the estimation of the BS-model is the value of the firm. We calculate the firm value as the sum of the values of its common shares (market price times the amount of shares outstanding), convertible bonds (market value of the bonds times the number of convertibles outstanding), and unobserved value of senior debt. The value of the senior debt is assumed to be equal to a constant amount over time and is calibrated by the Marquardt algorithm to provide the smallest possible deviation of the model price from the market price in the historical subsample.[15] In the original BS-model authors assume that convertible bondholders recover 2/3rd of the face value in case of bankruptcy. This implies that bankruptcy of a firm occurs as soon as the firm value net of senior debt drops to two-thirds of the value of the outstanding convertible bonds. In our study we allow this recovery proportion to be calibrated for all models.

We use the Crank-Nicolson (1947) finite difference algorithm for solving the corresponding partial differential equations and the Marquardt iterative procedure for finding the values of parameters that produce the smallest deviations of model prices from the market prices.

To calculate the convertible bonds values with the TF-model the following data are needed: bond issue date, trading date, risk-free rate, price of the underlying stock at the settlement date, maturity date, coupon rate, conversion ratio, call schedule, and the credit spread that reflects the credit rating of the issuer.[16] The only input needed for calculating prices with the TF-model that cannot be directly observed from the market is the credit spread. We

[15] We also estimated the version of the BS-model where we assumed senior debt to be zero. However, the model pricing errors tend to be larger than where we assume a constant positive value of the senior debt.

[16] Note that this model can also be estimated using the "cbprice" function of the Fixed Income Toolbox in Matlab.

use the average value of the credit spread for bonds that have the same credit ranking. Many of the bonds in our sample are issued by small firms, and therefore don't have credit ratings assigned. For companies that do not have credit ratings assigned, we assume a BBB rating. The average credit spreads are taken from the Canadian Corporate Bond Spread Charts published by RBC Capital Markets.

Prices are also calculated for the AFV-model. This model allows for different behavior of stock prices in case the firm defaults on its corporate debt. The partial default version assumes that the price of the underlying stock is partially affected by the firm's default on its bonds. The total default model assumes that the stock price jumps to zero when default takes place. We also assume that in case of a default convertible bondholders recover no value. In this study, we use the total default version of the AFV- model.[17]

V. Results

Our comparison of convertible bond pricing models is based on the scores that show the ability of the models to generate prices that are close to market prices. We will base our decision on several scores. The first and the most important score is the "Mean Absolute Deviation" (MAD). It is calculated as:

$$\text{MAD} = \text{average}\left[\text{abs}\left[\frac{\text{Market Price-Model Price}}{\text{Market Price}}\right]\right]$$

The MAD measures the pricing ability the best since it takes into account deviations from market prices from both sides. The second indicator is the "Mean Deviation" (MD). This is

[17] The results of the estimation of the partial default AFV-model are very similar to the totalpartial default AFV-model. These results are available on request from the authors.

calculated as the average deviation of the model price from the market price as a percentage of the convertible bond market price:

$$\text{MD} = \text{average}\left[\frac{\text{Market Price-Model Price}}{\text{Market Price}}\right]$$

The MD gives an idea of the average model over- or underpricing of the convertible bonds. Another indicator of model fit is the "Root of Mean Squared Error" (RMSE), which is calculated as:

$$\text{RMSE} = \sqrt{\text{average}\left[\frac{\text{Market Price - Model Price}}{\text{Market Price}}\right]^2}$$

The "Mean Absolute Deviation" and "Mean Deviation" scores assign the same weight to all errors. There is no additional penalty for the instances when the model price is far from the market price. The "Root of Mean Squared Error" score gives larger weights to large deviations.

We also calculate the percentage of forecasted model prices that fall within specific intervals around the market price as one of the measures of model pricing precision. We use ten, five and one-percent intervals around market prices for this purpose.

Table III2 provides rankings of the models based on the indicators mentioned above.

[Please insert Table III here]

Based on the results in Table III it can be concluded that the AFV-model has the best predictive power. This model shows the lowest values of mean absolute deviations. The value of the MAD is 1.7%. The AFV-model reports a slight overpricing of convertibles: the value of the MD is -0.43%. Almost ninety-nine percent of model errors are lower than 10% of bond market prices, 45.16% of errors are smaller than 1% of bond market value.

The second best model is the TF-model. This model has an average MAD of 1.74%.[18] On average, the TF-model overprices convertibles by -0.33%. This result is similar to that of Ammann, Kind, and Wilde (2003) who find that on average the TF-model overprices French convertible bonds by 3%. Slightly more than ninety-nine percent of the model predictions fall within 10% of market prices; 44.74% of the predicted pricing errors are less than one percent of the market price as compared to 45.16 per cent for the AFV-model.

The BS-model shows the largest MAD-score: 2.12%. The BS-model, on average, overprices the convertibles relative to the market by 1.23%; 97.99% and 37.05% of pricing errors are less than 10% and 1% of the market price respectively. The average pricing error is significantly different from zero for all three models. Therefore, we can reject the null hypothesis that the models have a mean zero pricing error.

Based on the Root of Mean Squared Error (RMSE) the most accurate model is the TF-model with an RMSE equal to 2.71. The second best is the AFC-model with RMSE of 2.78. The BS-model has the largest RMSE score of 3.11. Since the RMSE-score weights large pricing errors more heavily, the ranking of the models by the means of RMSE differs from the ranking based on the MD.

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[18] Since the sample used for estimating the BS-model is different from the sample for the other two models, we run robustness checks by comparing the results of the BS-model to the results of the other two models using the same subsample of firms. Even though the magnitudes of mispricing scores for the other two models ae marginally different, the newly calculated scores do not change the rankings of the models. These results are available on request from the authors.

Table IV3 provides the descriptive statistics for the mean deviations of the models' prices from the market prices both as the percentage of the market and the model prices.

[Please insert Table IVII here]

The BS-model shows the smallest range of errors.[19] The model' pricing ranges from underpricing the convertible securities by 5.8311% of the market price to overpricing by 13.930%; this creates a range of 19.7721%.[20] The pricing errors for the T-F model range from a negative 24% to a positive 9.6525%. For the Ayache, Forsyth, and Vetzal (2003) model, the errors span a range of 64.9358%.

It is also interesting to see whether there are convertible bond characteristics that affect the mispricing in a systematic way. In order to check for any such regularities we perform a regression analysis in which the pricing errors are regressed on characteristics of the convertible securities. These characteristics include the moneyness, measured as the ratio of the current market price to the conversion price (S/K), the annual historically observed volatility of the underlying stock (VOLAT), the convertible bond coupon rate (COUPON), and the remaining time to maturity of the convertible security (TMAT). We assume that the pricing errors are identically and independently distributed with a normal distribution that has an

[19] From Table 3 it can be observed that the BS-model has the smallest range of errors, 19.77% of market and 18.42%of model prices. All other models show much larger variations in errors. However, the BS-model uses only a subsample of firms, while the other models use the full sample of 64 firms. If the same subsample of 17 firms is used, the BS-model shows the smallest range of errors followed by the TF-model and the AFV-model. Detailed results are available from the authors on request.

[20] Errors are calculated as convertible market prices minus the corresponding model predicted prices.

expected value of zero and a variance equal to σ^2. Table V shows the results of the regressions where the dependent variable is the MAD. The results in this table help to find the variables that explain the 'precision' of the models.

[Please insert Table IV here]

As can be seen from Table V, the Mean Absolute Deviation statistically depends on the degree of the convertible bond moneyness for all three models in our study. On average, convertible securities that are deep in-the-money have smaller MADs than convertible securities that are at-the-money or out-of-the-money in TF and AFV-models while the result is reverse for BS-model.

The underlying stock volatility has a positive effect on the MADs for the TF- and AFV-models. This implies that convertible bonds with highly volatile underlying stocks are mispriced more heavily by all models but the BS-model.[21] This may happen because in the BS-model the firm volatility instead of the stock volatility is used for our calculations.

The AFV-model as well as the BS-model has smaller absolute deviations for the bonds with longer times to maturity however the coefficients are not statistically significant. The errors do not seem to depend on time to maturity of bonds in BS-model as the coefficient of 0.01 has neither statistical nor economic significance. This result is contrasting the result of King (1986) who finds heavier mispricing for the convertibles close to maturity in the BS-model. The accuracy improves for convertible bonds with shorter maturities in the TF-model.

[21] In the results section, a higher accuracy means lower absolute values of relative deviations.

The convertible bond coupon rates have a statistically significant positive effect on the size of the absolute values of the pricing errors for all but the AFV-model. Both the TF- and BS-models predict larger absolute deviations for the bonds with higherlower coupon rates. This effect is much more profound in the TFBS-model.

Table VI5 reports the regression results where the dependeant variable is the MD. The regression of the actual values of the pricing errors helps to find the variables that explain the 'direction' of mispricing, i.e. whether the convertibles are under- or overpriced.

[Please insert Table VI here]

From Table 5VI it can be seen that TF- and BS-models tend to overprice the bonds that are deeper in-the-money, the coefficient for TF-model is insignificant however. This result is in line with the results of King (1986), Carayannopoulos (1996), and Ammann, Kind, and Wilde (2003) who report a positive relation between overpricing and moneyness.

Volatility and time to maturity does not have any effect on the direction of pricing for the models in our study according to the results in Table VI. This result is in contrast to Ammann, Kind, and Wilde (2003) who find a larger underpricing for bonds with longer terms to maturity.

The convertible bond coupon rate hasrate has a positive and statistically significant effect on the MD for the TF-model. However, the BS-model on average overprices the convertible bonds with *higher* coupon rates; this effect is statistically insignificant for the AFV-model.

VI. Summary and conclusions

In this paper we compare the price prediction ability of one structural and two reduced-form convertible bond pricing models using actual market data on convertible bonds traded on the Toronto Stock Exchange. As opposed to other studies, we estimate all model parameters from *convertible* bond price series. This approach allows for the calculation of theoretical convertible bond prices even when the issuing firms have no ordinary debt outstanding or when parameters such as the credit spread or the dividend yield are not observable from market data. The final sample consists of 646 convertible bonds and spans the period from January 1, 2005 to April 28, 2006 for the two reduced-form models. For the Brennan and Schwartz (BS) model we use the subsample of 17 firms that have a simple capital structure that consists of equity, straight debt, and convertible debt only.

The results of our study show that both the Ayahche, Forsyth, and Vetzal (ATFV) model is the most accurate. On average, the mean absolute deviation, which is calculated as the absolute difference between the market and the model price expressed as a percentage of the market price, is 1.7% for the AFV-model, 1.74% for the TF-model, and 2.12% for the Ayache-Forsyth-Vetzal (AFV) Brennan-Schwartz model. This means that the model that requires more possibilities (non-discrete discount rate, possibility of total or partial default etc.), performs the best in terms of mean absolute deviations. It should be noted that, although the AFV-model requires the retunrns the lowest MAD, the TF-model requires fewer parameters and can be used for broader range of firms than the BS-model, since it can also be used for convertible bonds of companies with complex capital structures.

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Table I: Descriptive Statistics of the Convertible Bonds Sample Used
This table presents the descriptive statistics of the sample of 66 convertible bonds used in our study on the comparison of convertible bond pricing models. All the bonds in the sample are traded on the Toronto Stock Exchange. VOLAT refers to annualized historical standard deviation of the returns on the underlying stock; TMAT refers to the remaining time to maturity (in years) of the bonds as of December 1, 2005. COUPON refers to the convertible bond coupon rates. S/K refers to the ratio of average stock price during the forecast period to the conversion price.

	VOLAT	COUPON	TMAT	S/K
Minimum	0.13	0.05	1.25	0.3
Maximum	0.6	0.1	9.92	2.46
Average	0.25	0.072	4.92	1.06
Median	0.23	0.068	4.72	1.02
Standard Deviation	0.08	0.012	2.4	0.35

Table III: Calibrated model parameters' descriptive statistics

This table reports the descriptive statistics of the model parameters calibrated using the Marquardt algorithm. Parameter *d* is an implied dividend rate of an underlying stock; σ is an implied volatility of underlying stock, π is the excess calling cost as the proportion of the original call price used to proxy the presence of the soft call feature, *ρ (R* in AFV model*)* shows the price implied proportion of the bond value recovered in case of default. The model specific parameters are: for the AFV model α and γ are parameter of hazard function *p(S)*, where $p(S) = p_0^{'}(\frac{S}{S_0})^{\alpha}$, $\gamma = \frac{p_0^{'}}{S_0^{\alpha}}$; for the TF model r_c is an implied credit spread (in decimal form); for the BS-model *debt* denotes the calibrated value of the senior debt in the capital structure of a firm (in thousands of dollars). All these parameters are estimated using the historical sub-samples and later are fixed and used for out-of-sample price forecasts.

Ayache-Forsyth-Vetzal						
	α	γ	σ	d	π	R
mean	-2.10	0.09	0.33	0.21	0.09	0.03
median	-1.64	0.05	0.29	0.23	0.10	0.01
Min	-16.29	0.00	0.00	0.01	0.00	0.00
Max	2.03	0.48	0.94	0.30	0.30	0.34

Tsiveriotis-Fernandes					
	r_c	σ	d	π	ρ
mean	0.08	0.33	0.21	0.06	0.19
median	0.08	0.29	0.21	0.01	0.00
Min	0.00	0.00	0.02	0.00	0.00
Max	0.20	0.93	0.30	0.68	1.00

Brennan-Schwartz					
	debt	σ	d	π	ρ
mean	4049	0.43	0.26	0.10	0.10
median	2653	0.33	0.29	0.10	0.00
Min	0	0.07	0.03	0.00	0.00
Max	10000	1.00	0.30	0.30	0.51

Table III: Models' Mispricing Scores

This table reports the mispricing scores for all three models. The sample for the Ayache, Forsyth, and Vetzal (AFV) (2003) and Tsiveriotis and Fernandes (TF) (1998) models consists of 66 Canadian convertible bonds traded at the Toronto Stock Exchange. The subsample for the Brennan and Schwartz (BS) (1980) model consists of 17 firms, which have a capital structure that only consists of equity, straight debt, and convertible debt. Errors are calculated as convertible market prices minus the corresponding model predicted prices.

The mean deviation (MD) expressed in dollars refers to the average pricing error (in dollars) for the entire sample. The MD in the percentage form refers to the average error as a percentage of the convertible market price and is calculated as: $MD=\text{average}\left[\frac{\text{Market Price-Model Price}}{\text{Market Price}}\right]$

The Mean absolute deviation (MAD) refers to the average absolute error as a percentage of the convertible market price and is calculated as:

$$MAD=\text{average}\left[abs\left[\frac{\text{Market Price-Model Price}}{\text{Market Price}}\right]\right]$$

The Root of Mean Squared Error (RMSE) is calculated as the square root of mean squared error: $MRSE=\sqrt{\text{average}\left[\frac{\text{Market Price-Model Price}}{\text{Market Price}}\right]^2}$

The last three rows report the percentage of all predictions that fall within the defined range, i.e. "within 10%" means that the pricing error was less than 10 per cent of the market value.

	TF	BS	AFV
MD, $	-0.33	-1.23	-0.43
t-statistics	(-5.68)**	(-9.93)**	(-7.01)**
MD, %	-0.32	-1.18	-0.43
t-statistics	(-6.32)**	(-11.23)**	(-8.15)**
MAD, %	1.74	2.12	1.7
RMSE	2.71	3.11	2.78
Percentage of errors within 10% of market price	99.03	97.99	98.92
Percentage of errors within 5% of market price	95.29	90.34	95.55
Percentage of errors within 1% of market price	44.74	37.05	45.16

* and ** denote significance at the 5% and 1% levels, respectively

Table IV: Descriptive Statistics of Models' Over/Underpricing Errors

This table provides the descriptive statistics for the deviations of the observed market prices from the model prices expressed as a percentage of reference prices. MD_{market} refers to the errors as a percentage of the market price; MD_{model} is for the errors as a percentage of the model predicted prices.

The TF-model refers to the Tsiveriotis-Fernandes (1998) model. The AFV-model refers to the model of Ayache, Forsyth, and Vetzal (2003). The BS-model refers to Brennan-Schwartz (1980) model. The t-statistics are for the test of whether the average error is equal to zero.

	TF		AFV		BS	
	MD_{market}	MD_{model}	MD_{market}	MD_{model}	MD_{market}	MD_{model}
Mean	-0.32	-0.26	-0.43	-0.35	-1.18	-1.09
Median	-0.17	-0.17	-0.21	-0.21	-0.74	-0.74
Minimum	-23.55	-19.06	-39.25	-28.19	-13.93	-12.23
Maximum	9.65	10.68	25.68	34.55	5.83	6.20
Range	33.20	29.74	64.93	62.73	19.77	18.42
t-statistics	-6.32	-5.20	-8.15	-7.07	-11.23	-10.97

* and ** denote significance at the 5%, and 1% levels, respectively.

Table V: Regression Results for the Mean Absolute Deviations (MADs)

This table shows regression results of the models' Mean Absolute Deviations (MADs) on the ratio of the stock market price to the conversion price (S/K), the time to maturity of the convertible security (TMAT), the historically observed volatility of the underlying stock (VOLAT), and the convertible bond coupon rate. MADs are defined as the absolute difference between market the price and the model price divided by the market price.

The sample consists of 66 Canadian convertible bonds traded on the Toronto Stock Exchange. The start of the sample period depends on the issue date of the bond and starts either on January 1, 2005, July 1, 2005 or January 1, 2006. The sample period ends on April 28, 2006.

The TF-model refers to the Tsiveriotis-Fernandes model. The AFV-model refers to Ayache-Vetzal-Forsyth (2003) model. The BS-model refers to the Brennan-Schwartz (1980) model. The sample for the TF- and AFV-models consists of 66 firms traded on the Toronto Stock Exchange. For the estimation of the BS-model the subsample of 17 firms with the capital structure consisting of equity, straight debt, and convertible debt is used.

The t-values of the coefficient estimates are reported in parentheses.

Model	Intercept	S/K	VOLAT	COUPON	TMAT	R^2
TF	1.42	-0.94	4.49	6.43	-0.04	0.10
	(4.33)**	(-3.83)**	(3.33)**	(3.55)**	(-1.44)	
AFV	1.18	-0.63	5.50	-0.27	-0.02	0.05
	(2.58)**	(-3.22)**	(2.51)**	(-0.36)	(-0.51)	
BS	-6.82	0.47	-0.19	1.20	0.01	0.44
	(-5.16)**	(2.70)**	(-0.18)	(6.71)**	(0.15)	

* and ** denote significance at the 5% and 1% levels, respectively

Table IIIVI: Regression Results for the Mean Deviations

This table shows regression results of the Mean Deviations (MDs) on the ratio of the stock market price to the conversion price (S/K), the time to maturity of the convertible security (TMAT), the historically observed volatility of the underlying stock (VOLAT), and the convertible bond coupon rate (COUPON). MDs are defined as the market price minus the model price divided by the market price.

The sample consists of 66 Canadian convertible bonds traded on the Toronto Stock Exchange. The start of the sample period depends on the issue date of the bond and starts either on January 1, 2005, July 1, 2005 or January 1, 2006. The sample period ends on April 28, 2006. The TF-model refers to the Tsiveriotis-Fernandes model. The AFV-model refers to Ayache-Vetzal-Forsyth (2003) model. The BS-model refers to the Brennan-Schwartz (1980) model.

The sample for the TF- and AFV-models consists of 66 firms traded on the Toronto Stock Exchange. For the estimation of the BS-model the subsample of 17 firms with the capital structure consisting of equity, straight debt, and convertible debt is used. The t-values of the coefficient estimates are reported in parentheses.

Model	intercept	S/K	VOLAT	COUPON	TMAT	R^2
TF	-1.46	-0.06	1.67	9.70	0.01	0.03
	(-2.73)**	(-0.18)	(0.86)	(4.13)**	(0.12)	
AFV	-0.39	0.37	-3.65	1.02	0.08	0.02
	(-0.72)	(1.48)	(-1.48)	(1.01)	(1.49)	
BS	9.35	-0.70	0.21	-1.44	0.07	0.42
	(5.70)**	(-2.85)**	(0.13)	(-7.17)**	(0.80)	

* and ** denote significance at the 5% and 1% levels, respectively.

Appendix A: Characteristics of the Convertible Bonds Used in the Study

This table reports the main characteristics of the convertible securities used in our study. The sample consists of 66 Canadian convertible bonds traded on the Toronto Stock Exchange. The "Conversion ratio" shows the number of stocks that can be obtained in case of conversion for each 100 dollars of bond face value. The underlying stock volatility is expressed as the annualized standard deviation. In the call schedule the first number refers to the call price per 100 dollars of face value, the second number refers to the starting date of calling at this price. Calling continues until the next call date (if any) or until the maturity date if not specified otherwise.

Credit spreads are derived from the corporate credit rating using the 2005 Royal Bank of Canada relative value curves for Canadian corporate bonds. "Income trust" column reports whether the issuing entity was an income trust. "Call notice period" provides the minimum period between the calling announcement and actual call date. "Industry" specifies the area of specialization of the issuing firm.

Soft call lists conditions to be met before the bond can be called. For example, "20 days cumulative (consecutive) above 125% of conversion" means that the stock has to be traded for above 125% of its conversion price for at least 20 (consecutive) days in any given 30-day period before issuer can call the bond; "N" refers to the absence of the soft call condition for a given bond.

Asterisks (*) denote the firms with a simple capital structure consisting of equity, straight debt, and convertible debt. These firms are used in the estimation of the Brennan-Schwartz model.

Issuer (Symbol)	Issue Date	Maturity Date	Coupon, %	Conversion Ratio, per 100$ of par value	Underlying Stock Volatility	Credit Spread, basis points	Call schedule	Income Trust	Call notice period, days	Industry	Soft call
Advantage Energy AVN.DB.A	3-Jul	8-Aug	9	5.9	0.21	45	105 - 08/01/06, 102.5 -08/01/07	Y	30-60	Oil and gas	N
Advantage Energy AVN.DB.B	3-Dec	9-Feb	8.25	6.1	0.21	65	105 - 02/01/07, 102.5 -02/01/08	Y	30-60	Oil and gas	N
Advantage Energy AVN.DB.C	3-Jul	9-Oct	7.5	4.9	0.21	65	105 - 10/01/07, 102.5 -10/01/08	Y	30-60	Oil and gas	N
Advantage Energy AVN.DB.D	5-Jan	11-Dec	7.75	4.8	0.21	75	105 - 12/01/07, 102.5 -12/01/08	Y	30-60	Oil and gas	N
Agricore United AU.DB	2-Nov	7-Nov	9	13.3	0.3	65	100 - 12/01/05	N	30-60	Agriculture	N

Alamos Gold AGI.DB	5-Jan	10-Feb	5.5	18.9	0.48	65	100 - 02/15/08	N	30-60	Metals and mining	20 days cumulative above 125%
Alexis Nihon AN.DB*	4-Aug	14-Jun	6.2	7.3	0.14	128	100 - 06/30/08	Y	30-60	Real estate	20 days cumulative above 125%
Algonquin Power APF.DB*	4-Jul	11-Jul	6.65	9.4	0.17	69	100 - 07/31/07	Y	30-60	Utilities	20 days cumulative above 125%
Baytex Energy BTE.DB	5-Jun	10-Dec	6.4	6.8	0.26	115	105 - 12/31/08, 102.5 -12/31/09	Y	30-60	Oil and gas	N
Bonavista Energy BNP.DB	4-Jan	9-Jun	7.5	4.4	0.26	83	105 - 02/01/07, 102.5 -02/01/08	Y	30-60	Oil and gas	N
Bonavista Energy BNP.DB.A	4-Dec	10-Jul	6.75	3.5	0.26	105	105 - 12/31/07, 102.5 -12/31/08	Y	30-60	Oil and gas	N
Boyd Group BYD.DB*	3-Sep	8-Sep	8	11.6	0.6	45	105 - 09/30/04, 102.5 -09/30/05	N/A			
Calloway REIT CWT.DB	4-Apr	14-Jun	6	5.9	0.22	88	100 - 06/30/08	Y	30-60	Real estate	Previous day price above 125%
Cameco Corp CCO.DB*	3-Sep	13-Oct	5	4.6	0.4	82	100 - 10/01/08	N	30-60	Metals and mining	N
Can Hotel Inc. HOT.DB	2-Feb	7-Sep	8.5	10.4	0.15	65	100 - 03/01/05	Y	30-60	Real estate	20 days consecutive above 115%
Can Hotel Inc. HOT.DB.A	4-Nov	14-Nov	6	8.5	0.15	128	100 - 11/30/08	Y	30-60	Real estate	20 days consecutive above 125%
Chemtrade CHE.DB*	2-Dec	7-Dec	10	6.9	0.23	45	105 - 12/31/05, 102.5 -12/31/06	Y	30-60	Chemicals	Previous day price above 125%

Cineplex Galaxy CGX.DB*	5-Jul	12-Dec	6	5.3	0.27	75	100 - 12/31/08	Y	30-60	Media	20 days consecutive above 125%
Clean Power CLE.DB	4-Jun	10-Dec	6.75	9.8	0.34	69	100 - 06/30/07	Y	30-60	Utilities	Previous day price above 125%
Clublink LNK.DB*	Apr-98	8-May	6	5	0.18	65	100 - 03/15/03	N	30-60	Leisure	N
Cominar CUF.DB	4-Sep	14-May	6.3	5.8	0.15	88	100 - 06/30/08	Y	30-60	Real estate	20 days consecutive above 125%
Creststreet Power CRS.DB	5-Jan	10-Mar	7	10	0.19	65	100 - 03/15/08	Y	30-60	Utilities	Current price above 125%
Daylight Energy DAY.DB*	4-Oct	9-Dec	8.5	10.5	0.21	65	105 - 12/01/07, 102.5 -12/01/08	Y	N/A	Oil and gas	N
Dundee REIT D.DB	4-May	14-Jun	6.5	4	0.17	117	100 - 06/30/08	Y	30-60	Real estate	20 days consecutive above 125%
Dundee REIT D.DB.A	5-Apr	15-Mar	5.7	3.3	0.17	117	100 - 03/31/09	Y	30-60	Real estate	20 days consecutive above 125%
Esprit Energy EEE.DB*	5-Jul	10-Dec	6.5	7.2	0.23	65	105 - 12/31/08, 102.5 -12/31/09	Y	30-60	Oil and gas	N
Fort Chicago Energy FCE.DB.A	3-Jan	8-Jun	7.5	11.1	0.22	45	100 - 01/31/06	Y	30-60	Oil and gas	20 days consecutive above 125%
Fort Chicago Energy FCE.DB.B	3-Oct	10-Dec	6.75	9.4	0.22	55	100 - 12/31/06	Y	30-60	Oil and gas	20 days consecutive above 125%
Gerdau AmeriSteel Corp. GNA.DB*	Apr-97	7-Apr	6.5	3.8	0.44	65	100 - 04/30/02	N	30-60	Metals and mining	N

Harvest Energy HTE.DB	4-Jan	9-May	9	7.1	0.31	90	105 - 05/31/07, 102.5 -05/31/08	Y	30-60	Oil and gas	N
Harvest Energy HTE.DB.B	5-Jul	10-Dec	6.5	3.2	0.31	90	105 - 12/31/08, 102.5 -12/31/09	Y	40-60	Oil and gas	N
InnVest INN.DB.A	4-Mar	11-Apr	6.25	8	0.19	105	100 - 04/15/08	Y	30-60	Real estate	20 days consecutive above 125%
Inter Pipeline IPL.DB	2-Nov	7-Dec	10	16.7	0.22	45	100 - 12/31/05	Y	30-60	Oil and gas	20 days consecutive above 125%
IPC US REIT IUR.DB.U	4-Nov	14-Nov	6	10.5	0.21	90	100 - 11/30/08	Y	30-60	Real estate	20 days consecutive above 125%
IPC US REIT IUR.DB.V	5-Sep	10-Sep	5.75	9.1	0.21	65	100 - 09/30/08	Y	30-60	Real estate	20 days consecutive above 125%
Keyera KEY.DB*	4-Jun	11-Jun	6.75	8.3	0.24	65	100 - 06/30/07	Y	30-60	Oil and gas	20 days consecutive above 125%
Legacy Hotels LGY.DB*	2-Feb	7-Apr	7.75	11.4	0.2	65	100 - 04/01/04	Y	30	Hotels	20 days consecutive above 115%
Magellan Aerospace MAL.DB*	2-Dec	8-Jan	8.5	22.2	0.44	45	100 - 01/31/06	N	40-60	Aerospace	20 days consecutive above 125%
MDC Partners MDZ.DB	5-Jun	10-Jun	8	7.1	0.34	83	100 - 06/30/08	N	30-60	Marketing services	20 out of 30 consecutive days above 125%
Morguard Real Estate MRT.DB.A	2-Jul	7-Nov	8.25	10	0.13	45	100 - 11/01/05	Y	30-60	Real estate	20 days consecutive above 125%
NAV Energy NVG.DB	4-May	9-Jun	8.75	9.1	0.27	65	105 - 06/30/07, 102.5 -06/30/08	Y	30-60	Oil and gas	N

Northland Power NPI.DB*	4-Aug	11-Jun	6.5	8	0.25	69	100 - 06/30/07	Y	30-60	Utilities	20 days consecutive above 125%
Paramount Energy PMT.DB	4-Aug	9-Sep	8	7	0.22	65	105 - 09/30/07, 102.5 -09/30/08	Y	40-60	Oil and gas	N
Paramount Energy PMT.DB.A	5-Apr	10-Jun	6.25	5.2	0.22	65	105 - 06/30/08, 102.5 -06/30/09	Y	30-60	Oil and gas	N
Pembina PIF.DB.A	1-Dec	7-Jun	7.5	9.5	0.24	45	100 - 06/30/05	Y	30-60	Oil and gas	20 days consecutive above 125%
Pembina PIF.DB.B	3-Jun	10-Dec	7.35	8	0.24	55	100 - 06/30/06	Y	30-60	Oil and gas	20 days consecutive above 125%
Primaris REIT PMZ.DB*	4-Jun	14-Jun	6.75	8.2	0.2	117	100 - 06/30/08	Y	40-60	Real estate	20 days consecutive above 125%
Primewest Energy PWI.DB.A	4-Aug	9-Sep	7.5	3.8	0.24	83	105 - 09/30/07, 102.5 -09/30/08	Y	30-60	Oil and gas	N
Primewest Energy PWI.DB.B	4-Aug	11-Dec	7.75	3.8	0.24	105	105 - 12/31/07, 102.5 -12/31/08	Y	30-60	Oil and gas	N
Progress Energy PGX.DB	5-Jan	10-May	6.75	6.7	0.26	65	105 - 12/31/07, 102.5 -12/31/08	Y	30-60	Oil and gas	N
Provident Energy PVE.DB.A	3-Sep	8-Dec	8.75	9.1	0.21	65	100 - 01/01/07	Y	30-60	Oil and gas	N
Provident Energy PVE.DB.B	4-Jul	9-Jul	8	8.3	0.21	83	100 - 07/31/07	Y	30-60	Oil and gas	N
Provident Energy PVE.DB.C	5-Feb	12-Aug	6.5	7.3	0.21	105	100 - 08/31/08	Y	30-60	Oil and gas	N

Retirement Res REIT RRR.DB.B	3-Jul	11-Jan	8.25	8.1	0.25	95	100 - 07/31/07	Y	30-60	Real estate	20 days consecutive above 125%
Retirement Res REIT RRR.DB.C	5-Apr	15-Mar	5.5	8.8	0.25	95	100 - 03/31/09	Y	30-60	Real estate	20 days consecutive above 125%
Retrocom Mid-Market RMM.DB*	5-Jul	12-Jul	7.5	12.1	0.29	75	100 - 08/31/09	Y	30-60	Real estate	20 days consecutive above 125%
Rogers Sugar RSI.DB.A	5-Mar	12-Jun	6	18.9	0.24	75	100 - 06/29/08	Y	30-60	Food	20 days consecutive above 125%
Royal Host Real Estate RYL.DB	2-Feb	7-Mar	9.25	14.3	0.19	65	100 - 03/01/05	Y	30	Real estate	20 days consecutive above 125%
Summit Real Estate SMU.DB	4-Feb	14-Mar	6.25	4.7	0.21	88	100 - 03/31/08	Y	30-60	Real estate	20 days consecutive above 125%
Superior Propane SPF.DB	1-Jan	7-Jul	8	6.3	0.27	45	100 - 02/01/04	Y	30-60	Diverse	20 days consecutive above 125%
Superior Propane SPF.DB.A	2-Dec	8-Nov	8	5	0.27	45	100 - 11/01/05	Y	30-60	Diverse	20 days consecutive above 125%
Superior Propane SPF.DB.B	5-Jun	12-Dec	5.75	2.8	0.27	69	100 - 07/01/08	Y	30-60	Diverse	20 days consecutive above 125%
Superior Propane SPF.DB.C	5-Oct	15-Oct	5.85	3.2	0.27	88	100 - 10/31/08	Y	30-60	Diverse	20 days consecutive above 125%
Taylor NGL TAY.DB*	5-Mar	10-Sep	5.85	9.7	0.27	75	100 - 09/10/08	Y	30-60	Oil and gas	20 days consecutive above 125%