Who Holds Risky Assets?*

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Abstract Preference heterogeneity is a natural explanation for portfolio heterogeneity. In a dynamic environment in which preference heterogeneity is as extreme as possible, we show that intuition about the relationship between risk aversion and holdings of risky assets derived from static choice problems can be very misleading. In equilibrium, an agent with recursive utility who is infinitely risk averse over static gambles will hold a portfolio composed almost entirely of risky assets. Conversely, an agent with recursive utility who is risk neutral over static gambles will hold a portfolio composed almost entirely of risk-free assets. Moreover, there is no added compensation for holding the risky asset since equilibrium asset prices will appear to be generated by a risk-neutral representative agent. This counter-intuitive result highlights the relative roles of static risk preferences and deterministic substitution preferences in recursive utility. Since portfolio choice is fundamentally a decision about intertemporal consumption lotteries, both characteristics are important. Specifically, we show that the preference for the timing of the resolution of uncertainty plays a major role in allocation of investments between risky and risk-free assets. A strong preference for the late resolution of uncertainty translates into a strong preference for smooth consumption paths and, hence, a portfolio choice heavily skewed toward risk-free assets. This strong preference can exist even when the agent is risk neutral with respect to static gambles. Conversely, a strong preference for the early resolution of uncertainty translates into a strong preference for smooth utility which can be achieved even when portfolio choice heavily skewed toward risky assets.

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*This paper is preliminary – more so than most working papers – so comments are welcome and apologies for the rough edges.

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1 Introduction

Preference heterogeneity is a natural explanation for portfolio heterogeneity. The natural assumption, for example, is that more risk aversion dictates a portfolio more heavily weighted towards bonds. However, in a dynamic setting this intuition can be misleading. In particular, in a dynamic setting with recursive preferences (Epstein and Zin (1989), Kreps and Porteus (1978)), preferences involve both a static risk preferences and deterministic substitution preferences. Since portfolio choice is fundamentally a decision about intertemporal consumption lotteries, both characteristics are important. That is, a portfolio affects the variability of your consumption and the variability of future utility.

Models of recursive preferences are difficult to make tractable. This is especially so when considering preference heterogeneity (see the related paper of Backus, Routledge, and Zin (2007)). In this paper, we solve a model explicitly where preference heterogeneity is as extreme as possible. The result underscores how our intuition about the relationship between risk aversion and holdings of risky assets derived from static choice problems can be very misleading. In equilibrium, an agent with recursive utility who is infinitely risk averse over static gambles will hold a portfolio composed almost entirely of risky assets. Conversely, an agent with recursive utility who is risk neutral over static gambles will hold a portfolio composed almost entirely of risk-free assets. Moreover, there is no added compensation for holding the risky asset since equilibrium asset prices will appear to be generated by a risk-neutral representative agent. This counter-intuitive result highlights the relative roles of static risk preferences and deterministic substitution preferences in recursive utility. Since portfolio choice affects intertemporal consumption lotteries, both characteristics are important. Specifically, we show that the preference for the timing of the resolution of uncertainty plays a major role in allocation of investments between risky and risk-free assets. A strong preference for the late resolution of uncertainty translates into a strong preference for smooth consumption paths and, hence, a portfolio choice heavily skewed toward risk-free assets. This strong preference can exist even when the agent is risk neutral with respect to static gambles. Conversely, a strong preference for the early resolution of uncertainty translates into a strong preference for smooth utility which can be achieved even when portfolio choice heavily skewed toward risky assets.

The paper structure is straightforward. We consider the Pareto sharing problem in a setting with maximal preference heterogeneity. We can then address aggregation and asset prices and consider decentralization. We focus primarily on a setting with an aggregate income process that is iid. We also consider a income process that is persistent.

1.1 Related literature

To be added...

2 Recursive Utility Pareto Problem

A social planner allocates aggregate income y_t across two agents. The dynamics of income are characterized by a Markov process with transition probabilities $p(y_t|y_{t-1})$. Each agent has a recursive utility function defined by

$$\begin{aligned} U_t &= [(1-\beta)(c_t)^\rho + \beta\mu_t(U_{t+1})^\rho]^{\frac{1}{\rho}} \\ &\mu_t(U_{t+1}) = \left[E_t U_{t+1}^\alpha\right]^{\frac{1}{\alpha}} \end{aligned}$$

where preference parameters $\rho \leq 1$, $\alpha \leq 1$ and $0 < \beta < 1$ can differ across the two agents. The preferences aggregate consumption over time with a CES aggregator. The parameter ρ determines the inter-temporal rate of substitution over deterministic consumption at consecutive dates. The inter-temporal rate of substitution is $1/(1-\rho)$ and a small (negative) value of ρ indicates a strong preference to smooth consumption across time. Risk preferences are captured in the risk aggregator or certainty equivalence function μ. Over static gambles, the parameter α determines the risk aversion (the coefficient of relative risk aversion is $1-\alpha$). The preferences are homothetic. In addition, for convenience, utility is scaled to be in units of consumption. That is, if consumption is risk-free and constant $c_t = c$, $U_t = c$ (See Backus, Routledge, and Zin (2005) for more on recursive preferences.)

The marginal rate of intertemporal of substitution is given by

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(\frac{U_{t+1}}{\mu_t(U_{t+1})}\right)^{\alpha-\rho} \tag{1}$$

The marginal rate illustrates an important feature of recursive preferences. The MRS includes the growth rate in consumption. This is familiar from time-additive expected utility. In addition, recursive preferences include a term that compares the innovation in next-period's utility with its certainty equivalent. Note that this second term goes away when $\alpha_1 = \rho_1$ and preferences are the familiar time-additive expected utility.

The two-agent Pareto problem is a sequence of consumption allocations for each agent $\{c_{1,t}, c_{2,t}\}$ that maximizes the weighted average of date-0 utilities subject to the aggregate resource constraint which binds at each date and state:

$$\begin{aligned} \max_{\{c_{1,t},c_{2,t}\}} \lambda W_0 \quad &+ \quad (1-\lambda)V_0 \\ \text{s.t.} \;\; c_{1,t} + c_{2,t} \quad &= \quad y_t \quad \text{for all } s^t \end{aligned}$$

Note that even though each agent has recursive utility, the objective function of the social planner is not recursive. However, we can rewrite this as a recursive optimization problem following, Lucas and Stokey (1984), and Kan (1995):

$$\begin{aligned} J(V,s) \quad = \quad & \max_{c_1,c_2,V'(s')} \quad [(1-\beta)c_1^{\rho_1} + \beta\mu_{1,t}(J(V',s'))^{\rho_1}]^{\frac{1}{\rho_1}} \\ & s.t. \qquad [(1-\beta)c_2^{\rho_2} + \beta\mu_{2,t}(V')^{\rho_2}]^{\frac{1}{\rho_2}} \geq V \\ & \qquad\qquad c_1 + c_2 = y(s) \end{aligned} \tag{2}$$

where s is the current information set, $\mu_{i,t}$ is the certainty equivalent based on information s, and s' is next periods information set. The optimal policy involves choosing agent two's consumption, c_2, and a future utility V' (one for each future state, s') to maximize agent one's utility given an initial promise to agent 2 of utility, V. The promise serves as a law of motion for the state variable, V.

The solution to this problem is "perfect" or optimal risk sharing. Since we consider complete and frictionless markets, there is no need to specify the individual endowment process. This is in contrast to many papers that measure financial market fricitons by comparing, for example, the variance of an individual's consumption to the variance of an individual's income. In our setting, such a measure is not needed since allocations are, by construction, Pareto efficient.

3 Linear-Leontief Preferences

Recursive preferences allow the flexibility specifying risk aversion, α, and intertemporal substitution, ρ separately. The difference in these two parameters, $\alpha - \rho$, determines the preference for the timing of the resolution of uncertainty. It turns out that an example with two agents who differ wildly along this dimension is explicitly solvable. To develop a better understanding of how the Pareto problem in equation (2) works, consider preferences

$$\begin{aligned} W &= \min\{(c_1, E[W']\} \\ V &= (1-\beta)c_2 + \beta \min\{V'\} \end{aligned}$$

Agent 1 is risk neutral ($\alpha_1 = 1$) with perfectly inelastic deterministic substitution ($\rho_1 = -\infty$). In contrast, agent 2 has infinite risk aversion ($\alpha_2 = -\infty$) and perfectly elastic deterministic intertemporal substitution ($\rho_2 = 1$). Formally, we assume that $\beta_1 = \beta_2 = \beta$. However, note that the limit as the curvature parameter (ρ_1) goes to infinity eliminates the discount factor for agent 1. Similarly, as agent 2's curvature parameter (α_2) goes to infinity, the probabilities are eliminated. This leads to the "min" in their preferences. The agents, by construction, are very different. Agent 1 dislikes long-run risk, but tolerates short-run risk. Agent 2 is the opposite. In this extreme setting we can explicitly solve the Pareto problem to see how the consumption allocations vary over time. We can also, interestingly, solve for aggregate asset prices. Surprisingly (perhaps), this setting leads to a simple representative agent.

3.1 IID income endowment

To specify the recursive Pareto problem in this setting, let income be IID as $y_t = \bar{y} + \epsilon_t$ with $E\epsilon_t = 0$. The structure of the problem implies that no other features of the income process matter (the variance, for example).

$$\begin{aligned} J(y,V) &= \max_{c,V'} \min\{y - c, EJ(y', V')\} \\ \text{s.t.} \quad & V = (1-\beta)c + \beta \min\{V'\} \end{aligned}$$

That is choose agent two's consumption, c, and future utility, V' (one for each future state) to maximize agent one's utility from current consumption $y - c$ (applying the resource constraint) and future utility, $EW' = EJ(y', V')$. This is given an initial utility promise to agent two. The utility promised to agent two, V', defines the law of motion for the state variable.

We solve this problem with a guess and verification of the value function. Conjecture that the value function is linear (the obvious guess after you work out a few recursions).

$$J(y, V) \;=\; p_0 + p_y y - p_v V$$

for parameters (p_0, p_y, p_v) to be determined. We solve for these parameters and the allocations in three steps. (i) Since agent 2 has infinite risk aversion, utility promises are constant across states, $V' = \bar{V}'$. (ii) The level of current promised utility imply a consumption allocation of

$$c \;=\; \frac{V}{1-\beta} - \frac{\beta \bar{V}'}{1-\beta}$$

(iii) The optimization is, therefore,

$$p_0 + p_y y - p_v V = \max_{\bar{V}'} \min \left\{ y - \left(\tfrac{V}{1-\beta} - \tfrac{\beta \bar{V}'}{1+\beta} \right), p_0 + p_y \bar{y} - p_v \bar{V}' \right\}$$

Optimality occurs where the two quantities inside the min are equal (where current consumption equals future utility). This implies

$$\bar{V}' \left[p_v + \frac{\beta}{1-\beta} \right] \;=\; p_0 + p_y \bar{y} - y + \frac{V}{1-\beta}$$

Substitutes this relation into the Bellman equation and equate terms,

$$p_0 \;=\; \beta \bar{y}, \quad p_y \;=\; 1-\beta, \quad p_v \;=\; 1$$

This solution defines the Pareto frontier as

$$W \;=\; J(y, V) \;=\; (1-\beta)\bar{y} + \beta y - V$$

The controlled decision rules and laws of motion include

$$\begin{aligned} y - c &= \bar{y} - V + (1-\beta)(y - \bar{y}) \\ c &= V + \beta(y - \bar{y}) \\ W' &= \bar{y} - V + (1-\beta)(y - \bar{y}) + (1-\beta)\epsilon' \\ \bar{V}' &= V - (1-\beta)(y - \bar{y}) \end{aligned}$$

The exact solutions offer some insight into the optimal allocation rules. If there is an increase in income y, agent 2 consumes a fraction β of it and agent 1 consumes fraction $(1-\beta)$. In agent 1's case, optimality requires current consumption and expected future expected utility go up one-for-one; hence, the identical increase in W'. In agent 2's case, an increase in y leads to an increase in current consumption and a fall in future utility. This exploits the fact that agent 2 has infinite intertemporal substitution.

3.2 Persistent income endowment

The previous example shows how differences in attitude towards intertemproal risk effects the optimal income allocations. Therefore, it is interesting to see how the allocations behave if the endowment process follows a persistent process. Assume income y_t is AR(1) process

$$y' = (1-\varphi)\bar{y} + \varphi y + \epsilon'$$

where $\bar{y}$ is the unconditional mean, φ is the persistence parameter, and $E\epsilon' = 0$. Again, the structure of the problem implies that no other features of the process matter (the conditional variance, for example).

We solve the Pareto problem with the same steps as above (a bit more algebra). This leads to the value function parameters

$$p_0 = \frac{\beta(1-\varphi)}{1-\beta\varphi}\,\bar{y}, \quad p_y = \frac{(1-\beta)}{1-\beta\varphi}, \quad p_v = 1$$

The controlled decision rules and laws of motion include

$$
\begin{aligned}
y - c &= \frac{1-\varphi}{1-\beta\varphi}\bar{y} - V + \frac{1-\beta}{1-\beta\varphi}(y-\bar{y}) \\
c &= V + \frac{\beta(1-\varphi)}{1-\beta\varphi}(y-\bar{y}) \\
W' &= \bar{y} - V + \frac{(1-\beta)}{1-\beta\varphi}(y-\bar{y}) + \frac{(1-\beta)}{1-\beta\varphi}\epsilon' \\
\bar{V}' &= V - \frac{(1-\beta)(1-\varphi)}{1-\beta\varphi}(y-\bar{y})
\end{aligned}
$$

Comparing these results to the IID case above, shows the role of a persistent endowment process. Persistent income, $\varphi > 0$, increases the sensitivity of agent 1'consumption, $y - c$, to current income. The persistence of the shock to income makes it easier to increase agent 1's future utility, EW'. Recall, the inelastic intertemporal preferences of agent 1 require current and future utility to move one for one. The persistence reduces the sensitivity of agent 2's consumption, c, to current income.

3.3 Aggregation

Having solved for the income allocations, we can now describe asset prices in these economies. Given the extreme nature of the Linear-Leontief preferences ($\alpha_1 = 1$, $\rho_1 = -\infty$, $\alpha_2 = -\infty$, and, $\rho_2 = 1$), it is, perhaps, surprising that there exists a representative agent and that the representative agent has preferences in the class of recursive preferences. In fact, the representative agent in either economy (the income process does not matter) has linear preferences; that is $\tilde{\alpha} = \tilde{\rho} = 1$.

To start, consider the MRS of agent 2

$$
m_2' = \beta \left(\frac{c'}{c}\right)^{\rho_2 - 1} \left(\frac{V'}{\mu_2(V')}\right)^{\alpha_2 - \rho_2}
$$

The first term in brackets is one since $\rho_2 = 1$. Optimality implies agent 2's next period's utility is a constant, $V' = \bar{V}'$. Hence, the second term in brackets is one. This implies, the pricing kernel for agent 2 is β. For Agent 1, $\alpha_1 = 1$ implies

$$
m_1' = \beta \left(\frac{y'-c'}{y-c}\right)^{\rho_1 - 1} \left(\frac{W'}{\mu_1(W')}\right)^{\alpha_1 - \rho_1}
$$

$$= \quad \beta \left(\frac{(y' - c')EW'}{(y - c)W'} \right)^{\rho - 1}$$

Recall, that for agent 1, optimality requires that we equate current and future utility, hence $y - c = EW'$. Second, the Bellman equation implies $y' - c' = W'$ (a bit of algebra using the controlled laws of motion). This implies that the pricing kernel for agent 1 is, again, β. It is, of course, not surprising that optimal allocations imply $m_1' = m_2'$.

Despite the extreme preferences in this example, asset prices are equivalent to an economy with a representative agent with linear preferences, $\tilde{\alpha} = \tilde{\rho} = 1$, implying a pricing kernel of $m' = \beta$. This holds for either of the income processes we considered. In this case, the representative agent preferences are independent of the income process. This is a feature that is specific to this example. In general, as we explore below, the the representative agent preferences will reflect the preferences of the individual agents and features of the endowment process.

3.4 Income Distribution

An important characteristics in models with heterogeneous agents is the dynamic properties of the cross-section distribution of consumption (or wealth). If the two agents have consumption that grow at different rates, then the cross-section distribution degenerates to the point where one agent gets everything (in the limit). In the Linear-Leontief example does infinite risk aversion or inelastic substitution dominate the consumption cross-section? The answer is neither. Consider the case of an IID income endowment. The controlled laws of motion imply the consumption of both agents is a drift-less random walk. In addition, note that utility levels are also a random walk. In the case of an income endowment with persistence, the consumption and income levels of both agents are stationary.

3.5 Decentralized Equilibrium

Given the pricing kernel implied Pareto optimality, we can calculate an equity and bond price process. Note that the pricing kernel $m_{t+1} = \beta$ was identical across the two income processes we considered above. The one-period risk-free bond price is Bond price:

$$P_t^b = E_t m_{t+1} = \beta$$

which is constant for all t for any income process. Defining the equity price as the price of a claim to the aggregate income process implies

$$\begin{aligned} P_t^s &= E_t \sum_{j=1}^{\infty} \left[\prod_{i=1}^{j} m_{t+i} \right] y_{t+j} \\ &= \sum_{j=1}^{\infty} \beta^j E_t y_{t+j} \end{aligned}$$

Note that this price will depend on the income process through the conditional mean of the income process. To start, consider the iid income process.

In the case where aggregate income is iid, $E_t y_{t+j} = \bar{y}$ so the equity price is $P_t^s = [\beta/(1-\beta)]\bar{y}$, constant for all t. As noted above, the bond price is also constant, $P_t^b = \beta$. We can verify these equilibrium prices by finding a feasible portfolio strategy for each agent that achieves the Pareto optimal allocations.

Consider agent 2's portfolio choice in a stock-bond economy. The date-t budget constraint is

$$\theta_{t-1}^s (P_t^s + y_t) + \theta_{t-1}^b = c_t + \theta_t^s P_t^s + \theta_t^b P_t^b,$$

where θ_t^s and θ_t^b investments in equity and one-period bonds respectively. At the proposed equilibrium prices, the budget constraint becomes

$$\theta_{t-1}^s (P^s + \bar{y} + \varepsilon_t) + \theta_{t-1}^b = c_t + \theta_t^s P_s + \theta_t^b \beta.$$

Conjecture a solution for equity holdings of $\theta_t^s = \beta$ constant for all t. This implies

$$\beta\bar{y} + \beta\varepsilon_t + \theta_{t-1}^b = c_t + \theta_t^b \beta.$$

Recall that optimal consumption in this case satisfies

$$\begin{aligned} c_t &= V_t + \beta\varepsilon_t \\ &= V_{t-1} + (\beta - 1)\varepsilon_{t-1} + \beta\varepsilon_t \end{aligned}$$

$$= \quad V_0 + (\beta - 1)\sum_{j=1}^{t} \varepsilon_{t-j} + \beta\varepsilon_t,$$

Therefore any sequence of bond holdings such that

$$\beta\bar{y} + \theta^b_{t-1} - \beta\theta^b_t = V_0 + (\beta - 1)\sum_{j=1}^{t} \varepsilon_{t-j},$$

will support the optimal consumption allocation. In other words, the optimal bond holdings for Agent 2 solves the stochastic difference equation

$$\theta^b_t = \kappa_0 + \kappa_1\theta^b_{t-1} + \eta_t,$$

where $\kappa_0 = \bar{y} - V_0/\beta$, $\kappa_1 = 1/\beta$ and $\eta_t - \eta_{t-1} = [(1-\beta)/\beta]\varepsilon_{t-1}$.

Note that the asset holdings of Agent 1 can be found from the aggregate resource constraint. The bonds are in zero net supply and there is one unit of the stock that pays a per-period dividend of y_t. This implies that agent 1's equity holding is constant at $\theta_t = (1 - \beta)$ which, by construction, achieves agent 1's optimal consumption.

The interesting feature of this competitive equilibrium in our example is that the "infinitely risk averse" agent, Agent 2, holds almost all of the risky asset, $\theta^s_t = \beta$. The "risk neutral" agent, Agent 1, holds almost none, $1 - \beta$, of the risky asset. This example highlights the importance of the preference for the early resolution of uncertainty as a dimension of risk that can be captured with recursive utility.

In the case of perisitant income, $E_t y_{t+j} = \bar{y}(1 - \varphi^j) + \varphi^j y_t$. This implies an equity price of

$$\begin{aligned} P^s_t &= y_t \sum_{j=1}^{\infty} \beta^j \left[\bar{y}(1 - \varphi^j) + \varphi^j\right] \\ &= \left[\frac{\beta}{1-\beta} - \frac{\beta\varphi}{1-\beta\varphi}\right] + \frac{\beta\varphi}{1-\beta\varphi} y_t. \end{aligned}$$

The bond price is unchanged. We can construct portfolio holdings of the two agents to verify these equilibrium pricess in an analogous way.

[...Portfolio holdings in the AR(1) algebra to be added...]

4 Conclusion

With recursive utility, “risk aversion” (ie, α) is a statement about the risk aversion over one-step-ahead *utility* lotteries. “Resolution of uncertainty” (ie, $\alpha - \rho$) is a statement about the preference over *consumption* lotteries. In our example with extreme heterogeneity, person 2 (utility denoted V) wants his utility next period to be perfectly predictable. In addition, he doesn't care how much consumption fluctuates over time as long as utility is smooth. Person 1 (utility denoted W), in contrast, is unconcerned about utility fluctuations. However, their preference dictates that future consumption is perfectly predictable.

In the iid case, it is easy to see how Pareto optimal allocation satisfies these preferences. To see this, set β be close to 1 (of course $\beta < 0$ is required for things to be wll defined). In the iid-income case, person 2, infinitely risk averse (over utility lotteries), absorbs all consumption risk: $c_t = V_0 + \varepsilon_t$. The result is that by doing this, utility is always perfectly predictable: $V_t = V_0$. On the other hand, the person 1 who has an infinite preference for early resolution of uncertainty (again, over consumption lotteries) has perfectly predictable consumption every period: $y_t - c_t = \bar{y} - V_0$, which is constant. In the the iid case, the mean $\bar{y}$, is the only predictable component of future income. (It turns out that his utility is constant too, but that's an artifact he doesn't care about). A similar explanation holds for the persistent-income case (and when β is less not close to one).

The open question, of course, is does this intuition carry over to the case where preferences are not this extreme. For example, what do the Pareto allocations look like when one agent has a stronger preference for predictable utility and the other has a stronger preference for predictable consumption. In a related paper, Backus, Routledge, and Zin (2007) we explore these issues in a Gaussian/log-approximation setting.

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