

Value, Trading Strategies and Financial Investment of Natural Gas Storage Assets

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Abstract: By valuing a natural gas storage facility as a portfolio of real options, we study the value, trading strategies, technology, and financial investment of the storage for a risk neutral marketer to trade gas in an energy commodity exchange (spot market). The storage facility under consideration is a representative of highly deliverable peak load storages and lowly deliverable base load storages, with distinct deliverability restricted by physical injection and withdrawal constraints. Its operational flexibilities endow it with the real options which could be valued by the expected maximum profit earned from seasonal and daily trading. The primary focus is to optimize trading strategy to extract the embedded options value and assess the financial value of a storage contract to gain the whole or partial control of the facility over years. We show that the optimal trading policy will inject (withdraw) gas only when the market price exceeds (is exceeded by) the shadow price of the working gas in storage, less (plus) the marginal operating cost. Moreover, we explore the contingency of the optimal gas trading policy on the prediction of the spot prices by characterizing when it is optimal to *sell low and buy high*. Finally, we demonstrate that the analysis can be deployed to value *firm storage contract*, and related technological deployment and financial investment.

Key words: Asset pricing, investment, natural gas storage facility, energy asset, real options.

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1 Fundamentals of Gas Storage Assets

Since the inception of market deregulation in the U.S. the natural gas market has played an increasingly important role in the energy market. Gas is now traded on spot markets connected by a pipeline grid, and gas futures for delivery at Henry Hub and LA is traded at the New York Mercantile Exchanges (NYMEX). As a main fuel, natural gas consumption is therefore highly correlated with weather and triggers pronounced demand spikes in winter. However, because gas must be extracted from underground fields and moved through the pipeline system to end users, the gas supply is relatively inflexible. To accommodate higher winter demand, many storage facilities are built to ensure that gas is easily accessible. In addition, natural gas can be stored as LNG (Liquid Natural Gas), which requires a separate (and costly) infrastructure with storage being a major component. Storage has become even more important since LNG has emerged commercially. The natural gas storage service is unbundled from traditional gas sales and transportation services. Gas storage service is now offered as a distinct charged service and the trade of storage can be dynamically adjusted to price movements and other market conditions.

Gas storage facility allows the operator to store gas and vary inventory position in response to changed market conditions. Basically, there are two types of player associated with the natural gas storage facility: local distribution companies (LDCs) with the obligation to serve and energy marketers with the objective to maximize profitability. Energy marketer uses storage capacities to exploit arbitrage opportunities. Gas storage facilities are classified into depleted gas reservoirs, aquifers, and salt caverns and LNG, accounting for various operational characteristics: base (cushion) and working gas capacities, withdrawal and deliverability, injection capacity and cycling. The base gas is used to maintain adequate reservoir pressure and is generally never removed, while working gas storage is used to sell. Deliverability refers to the rate at which the reservoir can release gas from storage. The injection and withdrawal rates are a function of the pressure of the gas. The higher the pressure is, the higher the withdrawal rate gets, whereas at the same time the injection rate gets lower, and vice versa. Cycling refers to the number of times the working gas volumes can be injected and withdrawn in a year.

There are two basic uses for gas in storage: meeting seasonal demand requirements as insurance against unpredicted supply disruptions, and meeting peak load requirements. According to the role of storage, natural gas storage facilities can also be divided into two categories: *base load facilities* and *peak load facilities*. *Base load facilities* are capable of holding enough natural gas to satisfy long-term seasonal demand requirements. These reservoirs are larger,

but the deliverable amount is relatively lower. Thus, their capability of extracting natural gas underground each day is limited. However, they provide a prolonged and steady supply. Depleted gas reservoirs are the most common type of base load storage facility. *Peak load storage facilities* have high deliverability for short periods of time and can withdraw gas quickly. These facilities are designed to meet sudden, short-term demand increases, but cannot hold as much natural gas as base-load facilities. They can, however, deliver and replenish smaller amounts of gas in a shorter time, more quickly than base-load facilities. Peak load facilities can have turnover rates as short as a few weeks or days. Salt caverns are the most common type of peak load storage facility, although aquifers may be used to meet short-term demand as well. We refer to naturalgas.org¹ for an extensive introduction of related topics of natural gas markets and storage facilities.

The value of natural gas storage facilities consists of two components: intrinsic value and extrinsic value. The intrinsic value can be determined by forward price curves and seasonal price spread while the extrinsic value can be captured from the gas price volatility through dynamically optimizing injection and withdrawal operations. First, gas storage facilities are arbitrage mechanisms that allow the exploitation of the seasonal spreads. Natural gas storage facilities play a vital role in maintaining the reliability of the supply that is needed to meet high demand during the cold weather. By injection during non-peak season and withdrawal during peak season, gas storage facilities mitigate seasonal demand and price fluctuations. Seasonality is generally considered the main source of gas price profits which can be locked-in by using the forward or future contracts. For example, a typical July-December forward contract can reflect value of the calendar spread between July and December. Nevertheless, as short-term gas prices are highly volatile and unpredictable, natural gas storages can be used to match the changing market conditions through the operational flexibilities, and thus create real options value. For instance, even if the price observed today is high, it may jump to a higher level tomorrow, and storage can supply extra gas to grasp the further profit of unexpected, yet favorable price movement. Thus, it is crucial to exploit the extrinsic value from the operational flexibility of gas storage facilities. We refer to Maragos (2002) for a more comprehensive introduction of the intrinsic and extrinsic values of the storage and their financial implications.

The natural gas storage can be either valued under hedge strategy or spot trading (no-hedging) strategy. The value of storage increases with the amount of risk taken. The hedging strategy may reduce the risk but also adversely affects the flexibility of the storage facilities which leads

¹Naturalgas.org is developed and is maintained by the Natural Gas Supply Association to provide educational website covering a variety of topics related to the natural gas industry.

to undervaluation of the storage. Without consideration of the operational characteristics, the extrinsic value of a plain vanilla option can only provide the potential upsides of physical storage value. With consideration of the operational characteristics of physical storage assets, the spot trading (non-hedging) strategy gives the highest valuation (See Maragos 2002). In order to exploit the maximum return (value) from the operational flexibilities which is the real options value inherent in storage facilities, we will value the storage asset via the spot trading (operating) strategy with dynamic optimisations. Intrinsic value of storage facilities is easy to be captured by forward curves due to the predictable seasonal pattern in gas prices. The extrinsic value is embedded in the inherent operational flexibility of natural gas storage facilities. To capture the extrinsic value of natural gas storage, we need to incorporate the various operational characteristics of storage facilities and the dynamic nature of gas prices, in particular spot price volatility, mean-reversion and seasonality. Stochastic dynamic programming (DP) is a proper methodology to address the maximal real options value of natural gas storage assets in competitive markets by optimizing injection and withdrawal operations when taking into consideration of the physical, operational and contractual constraints of the storage facilities.

In this article, we explicitly incorporate operational characteristics associated with a natural gas storage into the real options valuation approach. The methodology that we employ is to formulate a continuous-time stochastic dynamic program (SDP) for the storage asset valuation problem. The spot price of natural gas is assumed to follow a class of general diffusion process in which the mean-reversion, jumps and seasonality features can be incorporated. Our model specifies how the storage value depends on operational constraints such as maximum injection/withdrawal rates (flexibility), the maximum working gas volume (effective capacity), as well as the dynamics of gas prices. We derive maximum storage value by dynamically deciding when gas is injected and withdrawn as time and market conditions change. We show that a marketer will only inject when the market price exceeds the marginal operating costs by an amount that reflects the value of the option to delay the delivery of the stored gas. Similarly, the marketer will only inject when the value of storing one more unit gas in storage exceeds the market price and the marginal operating costs. We also derive the structural properties of the model and characterize the optimal injection/withdrawal trading strategy with two state-dependent switch curves.

Then, we study the monotonicity of the optimal trading policy on spot prices and specify the conditions on price process under which the marketer will inject more or withdrawal less as price goes up. This reflects the *selling high and buying low* behavior of a speculator, which is

a common business experience.

We also extend our analysis to study the *summer-winter spread strategy* for a base-load storage facility in which injections and withdrawals are separated due to the physical constraints, such as depleted gas reservoirs. Finally, we study a *firm storage contract* and analyze the effect of changing the operational constraints on the optimal trading policy and the optimal capacity investment decision.

The remainder of this paper is organized as following. Section 2 illustrates how our paper relates to the existing literature. Section 4.1 presents a stochastic control model that we use to analyze optimal operations characteristics for gas storage facility without cycling constraints. Section 4.2 unveils the structure of the optimal policy by switching curves and their relation with gas market prices. Section 4.3 restricts the number of cycling periods to two in a year and re-derives conclusions attained in the model without cycling constraints. The structural properties that are derived in Sections 4.2 and 4.3 are exploited in Section 6 to study the investment in the storage facilities. After outlining the numerical scheme, we proceed to illustrative examples in Section 7. We conclude the paper in Section 8 by outlining several future projects.

2 Literature Review

The real options approach has attained significant importance in modern financial theory as a powerful tool in valuation and hedging physical assets. Real options create the bridge between investment analysis and daily operating and management decisions. The theory has been well developed since 1970's. We refer to Dixit and Pindyck [1994], Trigeorgis [1996], Smit and Trigeorgis [2004], and Amram and Kulatilaka [2005]) for a complete introduction of real options theory and its relationship with financial options. Real options create profound impact on energy sectors, where the existence of derivative instruments, such as futures, forward and options, allows a full exploitation of the options value embedded in the options of physical assets such as power plants, pipelines and oil fields. Ronn (2002) provides the various applications of real options to enhance capital budgeting decisions in energy area. The valuation of natural gas storage is also an application of real options approach.

The literature in valuation of natural gas storage consists can be divided into two major groups according to the methodologies applied. The first one is derivatives based approach which is widely used by practitioners. For instance, Gray and Khandelwal (2004) propose

a rolling intrinsic approach by using forward contracts. In their model, the storage holder captures the intrinsic value at the start of the contract and then dynamically adjusts his position in response to the changes of forward prices. Gas storage can be reduced a collection of *calendar spread call options*, paying out the spread between gas prices over seasons (Eydeland and Wolyniec 2003). A calendar spread call options gives the buyer the right to exercise a long position in a near month and a short position in a farther-dated month. The storage facility's rights of injection in summer and withdrawal in winter correspond to the long- and short-position of a calendar spread option. Pricing a calendar spread option captures the gas price spread between winter and the past summer intuitively, and provides a cost-effective way of computing the storage value and optimal operating strategy. *Virtual storage*, a notion introduced by Ahn et al. (2002), is a financial contract of passport options, while ignoring injection/withdrawal and delivery charge. Exclusive of the operational cost, the maximum revenue that can be extracted by a self-financing, dynamic trading strategy is much higher than the seasonal spread of the gas price and serves as an upper bound of storage value. However, the derivative pricing approach that includes calendar options and virtual storage ignores operational constraints, which renders the derived solution suboptimal and ad hoc. Ignoring physical and operational constraints may lead to either undervalue or overvalue the extrinsic value of the storage. Moreover, gas storage facilities are distinct from most energy assets, and thus cannot be priced by duplicating the valuation analysis of regular financial options, such as spread, swing and more (Eydeland and Wolyniec 2003).

Stochastic dynamic programming (SDP) or optimal stochastic control arises as a natural alternative approach to value the real options value of storage assets by explicitly modeling the physical and operational constraints. Our model also follows this fashion. However, existence works focus on computational issue when solving the stochastic control problems of the storage valuation. Various computation approaches are proposed to improve the computation efficiency and stability such as the finite difference approach (see e.g., Ahn et al. [2002], Thompson et al. [2003], Weston [2002], Chen and Forsyth [2006]), Monte Carlo simulation approach (see e.g., Ludkovski and Carmona [2005], Boogert and de Jong [2006]).

Our work differs from this existing literature accessing value of natural gas storage assets in that we focus on characterization of the optimal operating (injection and withdrawal) strategies and analyzing the effect of changing operational constraints and market dynamics on the optimal policies and the financial value of the storage asset (facility or contract). Through structural analysis of the storage model, we try to exploit the managerial insights when operating a gas storage facility and trading natural gas with it so as guidance when valuing the storage asset.

Our model also falls into the literature on inventory theory, especially the inventory control in fluctuating environments. Song and Zipkin (1993) consider replenishing stock for a single item inventory/production system within a Markovian environment. They show that the optimal inventory replenish policy is *state-dependent* based stock policy (no setup cost) or (s, S) policy (with setup cost). They also show that as the environmental state is stochastically monotone, the based-stock level is also monotone in the state. Li, Porteus and Zhang (2001) study the optimal operating strategy of a multi-plant capacitated manufacturing system in the presence of exchange risk. They also suggest that a switch policy contingent on exchange rates optimally selects production/inventory sites across countries. Under stochastic monotonicity some other mild conditions on the exchange rate, they show that the optimal switch threshold is monotone in the exchange rate. Our model extends this literature by incorporating the operational characteristics of storage facilities such as injection/withdrawal constraints and operating costs, and the dynamics of natural gas market which feature the storage model from usual inventory systems.

3 Calibration of Natural Gas Price Dynamics

We basically assume that the spot market of natural gas is liquid and complete and its ask-and-bid spreads of the gas prices are neglected. We consider that all of gas marketers to rent a storage facility are betting on the gas volatility of the future, whereas their trading transactions cannot affect the price movements of the gas.

We denote by S_t the *risk adjusted* price of natural gas at a time t prior to T , the end of planning horizon. $\{S_t : 0 \leq t \leq T\}$ is either single- or multi-dimensional continuous-time Markov process. Mathematically, the gas price S_t is just an abstract upper-semicontinuous strong Markov process. A general form of one dimensional price process can be described by the following stochastic differential equation (SDE):

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dz + \sum_{k=1}^K \chi_k(t, S_t, \xi_k)dN_k(t), \quad (1)$$

where $\mu(S_t, t)$ and $\sigma(S_t, t)$ are the drift and volatility functions of time t , z is a standard Brownian motion (Wiener process), $(N_1(t), \dots, N_K(t))$ is a K -dimensional compounded Poisson processes with joint intensities $(\lambda_1(t), \dots, \lambda_K(t))$ and jump sizes $(\chi_1(t, S_t, \xi_1), \dots, \chi_K(t, S_t, \xi_K))$ depending on independent random variables ξ_1, \dots, ξ_K . Brownian motion $z(t)$ and jump processes $(N_1(t), \dots, N_K(t))$ are defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = \{\mathcal{F}_t\}$ being its natural filtration generated with the Brownian motion and point processes. As gas

prices are spontaneously spiky, these random jumps are driven by various external factors, such as weather and rumors. At a price jump epoch, S_t is added with an amount $\chi_k(t, S_t, \xi_k)$ to become $S_t + \chi_k(t, S_t, \xi_k)$. For each $k = 1, \dots, K$, ξ_k is a random variable with distribution function $F_k(x)$ that represents the impact of the k -th factor on the gas price. For $k = 1, \dots, K$, ξ_k is a generic random variable and an independent drawer of k -th distribution F_k . In addition, these materialized drawers are independent across $1, \dots, K$. We assume that the coefficients introduced above satisfy standard regularity conditions that guarantee the existence and uniqueness of the strong solution of partial differential equation (1) (See Duffie [2001], Karatzas and Shreve [1988] or Økandal and Sulem [2005]).

Figure 1 exhibits through the futures prices on NYMEX from 2000 through 2007 the seasonality and mean-reversion with jumps of the price behavior of natural gas. The valuation of natural gas storage facility depends on price trend and momentum. Thus, we need to select an appropriate gas price model to capture these features.

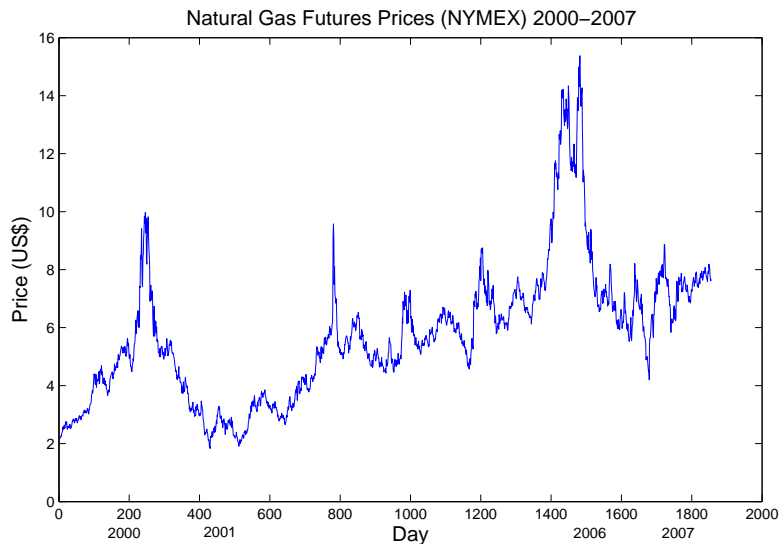


Figure 1: Natural Gas Futures Prices (NYMEX) 2000-2007

The random jumps presented in equation (1) describe the interferences of other marketing, political and environmental factors with the supply-and-demand balance. These jumps are characterized by a Poisson compounded stochastic process so that they happen at a sequence of Poisson arrivals with Bernoulli jump direction and exponential jump magnitude. One of the simplest forms of the jump-diffusion process with mean reversion is documented in Deng(1998)

and Eydeland and Wolyniec (2003) as follows:

$$dS_t/S_t = (\kappa(\theta - \log(S_t)) - \lambda\varpi)dt + \sigma_t dz_t + (\xi_t - 1)dN_t, \quad (2)$$

where the $\theta - \lambda\varpi - \sigma^2/(2\kappa)$ is the long-term mean of the logarithm of gas prices, N_t is a compound Poisson stochastic price process with arrival intensity λ , and the random magnitude $\xi_t - 1$ of the price jump with mean ϖ . Clearly, this model in (2) addresses spiky and mean-reverting features of the gas price. Then, the seasonality is captured by replacing θ in (2) with $\theta(t) = \theta_0 + \rho_0 \sin(4\pi(t - t_0))$, where $\theta_0 > 0$ is the equilibrium price without seasonality effect, ρ_0 is the semiannual seasonality parameter and t_0 is the seasonality centering parameters, denoting the time of semiannual peak of equilibrium price in summer and winter. The resulting adaptation that is named as the *Geometric Mean-Reverting model* with jumps is also adopted by Chen and Forsyth (2006). The model of the pure mean-reverting process with jumps is also often used:

$$dS_t = (\kappa(\theta - S_t) - \lambda\varpi)dt + \sigma_t dz_t + (\xi_t - 1)dN_t, \quad (3)$$

where the parameters are the same as for (2). Note that both (2) and (3) are the special cases of (1). There are many other special cases of (1), such as Brownian motion (BM) and Geometric Brownian motion (GMB).

The formulation of gas price is diverse, and the debate continues over which form is a good fit for gas prices, (Eydeland and Wolyniec [2003]). As our analysis fits a broad range of diffusion gas process models, we can sidestep the controversy of the gas model entirely. In addition, the analysis can extend from the one-factor diffusion process (1) to the two-factor diffusion process, such as the mean-reverting processes in Schwartz and Smith (2000).

4 Valuing a Peak-Load Natural Gas Storage Facility

In this section we formulate the operations control of a peak-load natural gas storage facility by a stochastic control model to maximize profit through operating the storage facility.

4.1 The Model

We consider a natural gas marketer to rent a peak-load storage facility with high deliverability and access to an energy commodity market. We assume that the marketer is risk neutral and can borrow and lend from a perfect capital market where the interest rate is r . As a price

taker, the marketer's primary objective is to maximize the expected profit earned running the storage facility in a finite planning horizon T , which is the duration of a lease contract. It is crucial to the marketer how the financial value associated with the contract should be assessed. The ongoing formulation is based on the operational characteristics of the facility and the marketer's project curves of the gas price.

The gas storage facility is operated in three core regimes – injection, withdrawal and storage. The gas is transferred into or out of the facility by a limited-capacity pipeline at a transfer rate that is determined by the operational characteristics. The maximum storage capacity of the facility is M millions of Btu (mmBtu). The injection and withdrawal rates are based on the pumping plant and the gas pressure in the facility. It is often the case that injection and withdrawal rates are the functions of gas load kept in underground storage facilities. We assume that injection rate a (mmBtu per unit of time) is bounded by a maximum injection rate $\alpha(x)$, and withdrawal rate w is bounded by a maximum rate $\beta(x)$, where x refers to the working gas load level in the facility. These operating characteristics differentiate the gas storage facility from other general inventory/production systems. We impose the following constraints on the operational characteristics of the facility exclusively. Other operating constraints, such as switching delays, have been curtailed for simplicity.

Assumption 1 *For any working gas load level x underground, the maximum injection rate $\alpha(x)$ is strictly decreasing and concave in x and the maximum withdrawal rate $\beta(x)$ is strictly increasing and concave in x . In addition, for any $\delta > 0$, $x + \alpha(x)\delta$ and $x - \beta(x)\delta$ are both strictly increasing in x^2 .*

Let q denote the amount of gas being injected ($q > 0$) into or released ($q < 0$) from the storage. We assume that the charge for injection/withdrawal operations, or the execution cost of the injection/withdrawal operations, $C(q, x)$, is linearly increasing in q for inventory level $q \geq 0$ and decreasing in q for $q \leq 0$. Typically, we assume that $C(q, x) = c_i q^+ + c_w q^-$.

In addition, certain amount of gas is burned to inject or withdraw gas and resulted in the fuel loss that accounts for no more than 2 percent of the whole gas amount that is injected or withdrawn. Let $L(q, x)$ denote the amount of gas lost when releasing $-q$ ($q < 0$) units from or injecting q ($q > 0$) unites into the gas storage, which holds x units of gas load. More specifically, we assume that the fuel loss is proportional to the injection/withdrawal rates q and

²If for a small $\delta > 0$, $1 - \beta'(0)\delta > 0$, then the derivative of $x - \beta(x)\delta$, $1 - \beta'(x)\delta \geq 1 - \beta'(0)\delta$ for $0 \leq x \leq M$, and thus $x - \beta(x)\delta$ is strictly increasing in x . An analogous argument ensures that for a small $\delta > 0$, $x + \alpha(x)\delta$ can be a strictly increasing function of x .

independent of the inventory level x , i.e., $L(q, x) = L(q) = \rho_i q^+ + \rho_w q^-$, where $\rho_i, \rho_w \in [0, 1)$.

Meanwhile, keeping gas underground incurs a holding cost for maintenance and various operations, which is counted per unit of time by $h(x, s)$ on the current gas load x and spot price s . Without loss of generality, we assume that $h(0, s) = 0$, $h(x, s)$ is increasing and convex in x for $x \geq 0$, non-decreasing in s and *supermodular* in (x, s) .

In practice, there are operational delays for switching operational regimes. Most studies ignore the delays and associated costs, while others convert operational switching lead times in a tractable manner into the warmup periods during which injection or withdrawal is undertaken at specified transition rates. We follow most of papers in the literature to ignore switch delays for simplicity.

Let x_0 denote the gas base load level at the beginning of the planning horizon, and X_t the working gas load level at a time t . As the facility is required to maintain its underground gas load at least x_0 while operating, underground gas load $X_t + x_0 \geq x_0$ or $X_t \geq 0$ for $t \geq 0$. At t , we let a_t and w_t denote injection and withdrawal rates at a time t , implying $q_t = a_t - w_t$. These two rates depend on the gas load level X_t and are subject to $a_t \in [0, \alpha(X_t)]$ and $w_t \in [0, \beta(X_t)]$. Thus, $q_t \in [-\beta(X_t), \alpha(X_t)]$. When rate q_t is given, the dynamics of the facility's gas load are given by

$$dX_t = q_t dt, \tag{4}$$

where dX_t denotes an instant change in gas load level X_t .

We let $\pi(t, X_t, S_t, q_t)$ denote the rate of instant payoff at time t for any (X_t, S_t, q_t) . Explicitly,

$$\pi(t, X_t, S_t; q_t) = \begin{cases} -S_t q_t (1 + \rho_i) - c_i q_t - h(X_t, S_t) & \text{Injection } (q_t > 0), \\ -h(X_t, S_t) & \text{Do nothing } (q_t = 0), \\ -S_t q_t (1 - \rho_w) - c_w q_t - h(X_t, S_t) & \text{Withdrawal } (q_t < 0). \end{cases}$$

At the end of the contract period, the storage contract is settled and the gas marketer will hand over the storage facility to its owner. At settlement the contract holder receives a potential penalty, denoted by $\nu_T(x, s)$. As the storage owners usually specify in contracts that the inventory level at the end of the contract period should be kept in the same level as the initial inventory level $X(0)$. Otherwise, the marketer should pay for the over-use or under-use as a penalty. Typically, we assume that $\nu_T(x, s)$ is concave in x for any s and supermodular in (x, s) . A common form of the penalty function is

$$\nu_T(x, s) = s[(1 - K_1)(x - X(0))^+ - (1 + K_2)(X(0) - x)^+],$$

where K_1 and K_2 are the penalty coefficients as proportions of the spot price and $0 \leq K_1 \leq K_2$. This function infers that the marketer has to return the overused gas by paying the spot price and an additional penalty proportional to the spot price, and vice versa. Clearly, $\nu_T(x, s)$ is concave in x for any fixed s and supermodular in (x, s) .

The expected profit from dynamically loading or releasing underground gas over a finite planning horizon is maximized by the approach of a continuous-time Markov decision process (MDP). The state of the process is denoted by (t, x, s) , where t refers to a time, x refers to the gas load level, and s refers to the gas price at t . Let \mathcal{U} be the set of *admissible strategies*, which are consist for all non-anticipative strategies. Non-anticipative describes decisions that are made on the historical information about the operating and trading process of the storage facility. For any admissible $u \in \mathcal{U}$, we let u_t denote the set of admissible decisions of u at time t , which specifies a decision with respect to state (t, x, s) . Following an admissible u , we let X_t^u denote the gas load at a time t , which together with gas price S_t forms a dynamic triple (t, X_t^u, S_t) that results from conducting u . In particular, $u_t = q_t^u$, where q_t^u refers to the injection rate or withdrawal rate of u at t , respectively. We let $J^u(t, x, s) \in \mathcal{D}$ denote the expected profit that is cumulatively gained from t through T under u , where $\mathcal{D} = [0, T] \times [0, M] \times [0, +\infty)$. Explicitly,

$$J^u(t, x, s) = E \left[\int_t^T e^{-r(\tau-t)} \pi(\tau, X_\tau^u, S_\tau; q_\tau^u) d\tau + e^{-rT} \nu(X_T^u, S_T) \middle| S(t) = s \right],$$

where the first term of the right-hand side integral counts the cash flow out of trading gas of the facility, the second and third terms count the corresponding operation cost of loading or unloading gas, the fourth term counts the holding cost of the gas in the storage, and the last term is the present value of the terminal revenue.

The problem of interest that is stated in the framework of the optimal stochastic control is to find out $u^* \in \mathcal{U}$ to attain

$$J^{u^*}(t, x, s) = J(t, x, s) = \sup_{u \in \mathcal{U}} J^u(t, x, s)$$

for any state (t, x, s) , subject to the dynamic constraints of

$$\begin{aligned} dS_t &= \mu(S_t, t)dt + \sigma(S_t, t)dz + \sum_{k=1}^K \chi(t, S_t, \xi_k) dN_k, \quad S_0 = s, \\ dX_t^u &= (q_t^u + L(q_t^u, X_t^u))dt, \quad q_t^u \in [-\beta(X_t^u), \alpha(X_t^u)], \end{aligned}$$

and terminal and boundary conditions

$$J^u(T, x, s) = v_T(x, s), \tag{5}$$

$$\lim_{s \rightarrow \infty} J_{ss}^u(t, x, s) \rightarrow 0, \text{ and } \lim_{s \rightarrow 0} J_{ss}^u(t, x, s) \rightarrow 0, \tag{6}$$

where terminal function $v_T(x, s)$ is an increasing and concave function of x for each fixed s , and $J_{ss}^u(t, x, s) = \frac{\partial^2 J^u(t, x, s)}{\partial s^2}$ is the second partial derivative of $J^u(t, x, s)$ with respect to spot price s . Limiting equations (6) imply that when the spot price is extremely small or large, the growth of $J^u(t, x, s)$ is asymptotically linear with respect to spot price s .

The ensuing theorem addresses that optimal value-to-go function $J(t, x, s)$ satisfies the Hamilton-Jacobi-Bellman equations and ensures the optimality of the model. In this regard, let \mathcal{L} denote an infinitesimal operator that is given by

$$\mathcal{L}\psi(t, x, s) = \psi_t(t, x, s) + \frac{1}{2}\sigma^2(s, t)\psi_{ss}(t, x, s) + \mu(s, t)\psi_s(t, x, s).$$

on a sufficiently smooth function $\psi(t, x, s)$, where

$$\psi_t(t, x, s) = \frac{\partial\psi(t, x, s)}{\partial t}, \quad \psi_x(t, x, s) = \frac{\partial\psi(t, x, s)}{\partial x},$$

and

$$\psi_s(t, x, s) = \frac{\partial\psi(t, x, s)}{\partial s}, \quad \psi_{ss}(t, x, s) = \frac{\partial^2\psi(t, x, s)}{\partial s^2}.$$

In addition, define another differential operator as

$$\tilde{\mathcal{L}} = \mathcal{L}\psi(t, x, s) - r\psi(t, x, s) + \sum_{k=1}^K \lambda_k [E\psi(t, x, s + \chi_k(t, s, \xi_k)) - \psi(t, x, s)],$$

on the same ψ by taking into account jumps and interest rate r .

Theorem 1 *Optimal value-to-go function $J(t, x, s)$ satisfies the following*

$$0 = \tilde{\mathcal{L}}J(t, x, s) - h(x, s) + \max_{q \in \mathcal{Q}(x)} \{-s(q + L(q)) - C(q, x) + qJ_x(t, x, s)\} \quad (7)$$

$$= \tilde{\mathcal{L}}J(t, x, s) - h(x, s) + \max_{q \in \mathcal{Q}(x)} \{[s(1 - \rho_w) - c_w - J_x(t, x, s)]q^- + [J_x(t, x, s) - s(1 + \rho_i) - c_i]q^+\}, \quad (8)$$

with terminal and boundary conditions (5) and (6).

Proof. See Appendix. □

It is appealing to interpret the ownership of the storage under the optimal control of (8) as the entitlement of real options to inject gas to or withdraw gas from the storage facilities. The storage contract can be deemed as a portfolio of American-type call and put options and its capability of injecting or withdrawing gas can be interpreted as holding call or put options exercisable in the exchange. Given any (t, x, s) , the strike price of an injection options

is $J_x(t, x, s) - c_i - \rho_i s$ that equals the rate of convenience yield less injection cost (exercise cost) and it is optimal to exercise the options if $s < J_x(t, x, s) - c_i - \rho_i s$. Similarly, the strike price of withdrawal options is $J_x(t, x, s) + c_w + \rho_w s$ and it is optimal to exercise it if $s > J_x(t, x, s) + c_w + \rho_w s$.

Thus, the optimal control can be better understood by exercising options, as follows. We divide the states of underlying MDP into three regions

$$\begin{aligned} \mathbf{IR} &= \{(t, x, s) \in \mathcal{D} : J_x(t, x, s) > s(1 + \rho_i) + c_i\}, \\ \mathbf{WR} &= \{(t, x, s) \in \mathcal{D} : J_x(t, x, s) < s(1 - \rho_w) - c_w\}, \\ \mathbf{NT} &= \{(t, x, s) \in \mathcal{D} : s(1 - \rho_w) - c_w < J_x(t, x, s) < s(1 + \rho_i) + c_i\}, \end{aligned}$$

where **IR** stand for the region of exercising injection (buying) options, **WR** for exercising withdrawal (selling) options, and **NT** for doing nothing, respectively.

The HJB equation (8) reveals that the optimal trading quantity rate is up to the physical delivering limit of the contract or storage

$$q^*(t, x, s) = \begin{cases} \alpha(x) & \text{if } (t, x, s) \in \mathbf{IR} \\ -\beta(x) & \text{if } (t, x, s) \in \mathbf{WR} \\ 0 & \text{if } (t, x, s) \in \mathbf{NT}. \end{cases}$$

4.2 Optimal Trading Strategy

In this part of the section, we engineer optimal trading strategy by the concavity of the optimal value-to-go function $J(t, x, s)$. The analysis needs to be conducted primarily in the context of a discrete-time Markov decision process.

4.2.1 A Discrete-Time MDP

When gas transitions are scheduled on a daily basis, we can deploy a discrete-time Markov decision process to optimize trading strategies. Formally, we calibrate the length of a period by a positive number of $\delta > 0$ days such that $T = N\delta$ and the contract duration consists of N periods of δ days. A regular specification of δ is one day, and T is thus equal to N days. Dividing the entire planning horizon $[0, T]$ into N equal-length periods

$$[0, \delta], (\delta, 2\delta], \dots, ((N-1)\delta, T]$$

constitutes a time grid in which the marketer's period-to-period decisions are made in the beginning of each period. Let $S_n = S_{(n-1)\delta}$ be the gas spot price in the n -th period and

$f_n(x, s)$ the optimal value function from the n -th period to the last period, with x being the gas load level in the beginning of the n -th period and $s = S_n$ being the spot price of that period. The optimality equations of the underlying Markov decision process are given by

$$f_n(x, s) = \max_{q \in [-\beta(x), \alpha(x)]} \left\{ - (q + L(q))s\delta - C(q)\delta + e^{-r\delta} E[f_{n+1}(x + q\delta, S_{n+1}) | S_n = s] - h(x, s)\delta \right\}. \quad (9)$$

The first term inside the braces in the right-hand side of (9) is $[(1 - \rho_w)s - c_w]q^- \delta$ when $q < 0$ as the net profit of withdrawing $q^- \delta$ units of gas in the period, while it is $[(1 + \rho_i)s + c_i]q^+ \delta$ when $q > 0$ as the cost of injecting $q^+ \delta$ units of gas. In addition, $e^{-r\delta}$ is the interest factor used to discount the revenue received in the $(n + 1)$ -st period and onward to the beginning of the n -th period. The last term inside the braces, $h(x, s)\delta$, is the cost of holding x units of gas in the storage over the current period.

Let $y = x + q\delta$ be an level of gas load in the end of the period, accounting for a result of trading $|q|$ units of gas in the period. In view of the physical restriction that $q \in [-\beta(x), \alpha(x)]$ $y \in [x - (1 - \rho_w)\beta(x)\delta, x + (1 + \rho_i)\alpha(x)\delta]$. Let $\mathcal{Q}(x) = [x - (1 - \rho_w)\beta(x)\delta, x + (1 + \rho_i)\alpha(x)\delta]$ denote the feasible interval for y , then the optimality equation (9) can be rewritten as

$$f_n(x, s) = \max_{y \in \mathcal{Q}(x)} \left\{ - Q(y - x)s\delta - C(Q(y - x))\delta + e^{-r\delta} E[f_{n+1}(y, S_{n+1}) | S_n = s] \right\} - h(x, s)\delta.$$

It is shown by the next lemma that the concavity of the $f_{n+1}(x, s)$ in x for fixed s can be preserved by $f_n(x, s)$.

Lemma 1 *Suppose that $f_{n+1}(x, s)$ is a concave function of x for any s . Then, $f_n(x, s)$ is also a strictly concave function of x for any fixed s .*

Proof. See Appendix. □

The preceding lemma is used to show that the concavity can be preserved by the optimal revenue function of the discrete version of the setting.

Theorem 2 *Suppose that terminal value function $v_T(x, s)$ is concave in x for any s . Then, for any $n \geq 1$, $f_n(x, s)$ is concave in x for any s .*

Proof. As $f_{N+1}(x, s) = v_T(x, s)$ is concave in x for any s , in view of Lemma 1, by a backward induction, we can deduce that $f_n(x, s)$ is concave in x for any s and n . □

The above theorem stands in a realistic hypothesis that pipeline grid systems and pump plants connected with the storage limit the flexibility of transferring gas and therefore, a plan of transferring gas cannot be completed abruptly. The concavity of the optimal revenue account for a common phenomenon that although the optimal revenue of keeping gas underground grows with the gas load, its marginal increase is decreasing. Many factors give rise to this phenomenon. One of main factors is the considerable increase in the physical restriction and cost with more gas to be kept in the storage facility.

4.2.2 The Bang-Bang Structure of the Optimal Policy

We are ready to prescribe the bang-bang structure of optimal trading policy for the continuous-time model. For simplicity, we transfer the structural properties from the discrete-time optimal policy to the continuous-time optimal policy.

Suppose that there is a sequence of $\{\delta_n : n \geq 1\}$ that converges to zero as $n \rightarrow \infty$, and for any n , δ_n can be used for discretizing $[0, T]$ such that $\frac{T}{\delta_n} = N_n$ is an integer. For any $t \in [0, T]$, let $m_n(t)$ denote an integer that is defined by

$$m_n \delta_n = \min\{k : k \delta_n \geq t\}$$

and represents the subinterval corresponding to δ_n that contains t . It follows that $t = \lim_{n \rightarrow \infty} m_n \delta_n$ and

$$\lim_{n \rightarrow \infty} f_{m_n}^{\delta_n}(x, s) = J(t, x, s)$$

where $f_k^{\delta_n}(x, s)$ is the optimal value function related to δ_n over periods k to N_n . As the limiting process preserves the concavity, the following theorem formally addresses the main result of this paper.

Theorem 3 $J(t, x, s)$ is strictly concave in x for any fixed (t, s) .

Proof. See Appendix. □

In line with Theorem 3, for fixed (t, s) , as $J(t, x, s)$ is a concave function of inventory level x , marginal revenue $J_x(t, x, s)$ is decreasing in x . This structures the optimal loading and releasing strategies of the facility. The following theorem characterizes the structure of the optimal trading strategy.

Theorem 4 For any fixed (t, s) , there is an inject-up-to gas load level $I(t, s)$ and a withdraw-down-to level $W(t, s)$, which are given by

$$I(t, s) = \inf \left\{ x \in [0, M] : J_x(t, x, s) \leq (1 + \rho_i)s + c_i \right\}, \quad (10)$$

and

$$W(t, s) = \inf \left\{ x \in (0, M] : J_x(t, x, s) \leq (1 - \rho_w)s - c_w \right\}. \quad (11)$$

Further, $I(t, s) \leq W(t, s)$. It is optimal to inject gas at rate $\alpha(x)$ if and only if $x < I(t, s)$, and to withdraw gas at rate $\beta(x)$ if and only if $x > W(t, s)$, and to do nothing otherwise. That is,

$$q^*(t, x, s) = \begin{cases} a(t, x, s) & \text{if } x \in [0, I(t, s)) \\ 0 & \text{if } x \in [I(t, s), W(t, s)] \\ -w(t, x, s) & \text{if } x \in (W(t, s), M] \end{cases} .$$

Theorem 4 shows that the optimal operating/trading policy of the natural gas storage is of state-dependent “bang-bang” type. That is, give each elapsed time and spot price pair (t, s) , there are two thresholds such that the optimal policy is to inject with maximal injection rate as the inventory level is below the lower thresholds and to withdrawal with maximal withdrawal rate as the inventory level is above the higher thresholds and do nothing as the inventory level is on or between them.

According to Theorem 4, surfaces $(t, I(t, x), s)$ and $(t, W(t, x), s)$ slice \mathcal{D} into three regions. In particular, $(t, I(t, x), s)$ and $(t, W(t, x), s)$ stand for the intersection of **IR** and **NT**, and that of **NT** and **WI**, respectively.

Remark 1 That the policy of “bang-bang” structure is optimal relies on the linearity of injection/withdrawal cost rate and the concavity of the optimal revenue function. If the linearity fails to hold, the policy with bang-bang structure may not be optimal.

4.3 Valuing a Base-Load Natural Gas Storage Asset

This section deals with a sort of *summer-winter* trading strategies for a base-load storage facility such as a depleted reservoir. Due to relatively low injection and deliverability rates, depleted reservoirs are normally used for seasonal storage rather than peak-shaving storage. It is typically single-cycle per year facility which turnovers twice annually. It buys gas cheap in summer for injection and sells it expensive in winter by withdrawing gas underground. The injection season is from April to October and the withdrawal season is from November to March

in the following year. During the injection season, the storage marketer observes the spot price and dynamically determines at what time and how much to purchase from the spot market and inject into the storage facility; during the withdrawal season, the marketer dynamically sells the gas from the facility back to the spot market. This model can be regarded as a special case of the peak-load storage model studied in Section 4.1. For simplicity we outline the main results for a contract effective duration $[0, T]$, which is a fiscal year for the marketer running from the April of one year through to the March of the following year.

Let $\tau \in [0, T]$ be the threshold that divides the whole duration into the summer and winter. The physical constraints on gas transition discussed in Section 4.1 are also applicable for the base load facility. However, the facility is filled during the summer months $[0, \tau]$, and gas is extracted during the winter months, $(\tau, T]$.

Thus, the dynamic optimization framework that we deployed for the peak-load facility can be modified for the base-load facility. As a result, we attain a simplification of Theorem 1 to maximize the revenue of the base load storage by injecting gas in summer season and withdrawing it in winter.

Theorem 5 *Optimal value-to-go function $J(t, x, s)$ satisfies the following:*

for $t \in [\tau, T]$,

$$0 = \tilde{\mathcal{L}}J(t, x, s) - h(x, s) + \max_{q \in [0, \alpha(x)]} \{-s(1 + \rho_i)q - C(q, x) + qJ_x(t, x, s)\}, \quad (12)$$

and for $t \in [0, \tau]$,

$$0 = \tilde{\mathcal{L}}J(t, x, s) - h(x, s) + \max_{q \in [0, \beta(x)]} \{s(1 - \rho_w)q - C(q, x) - qJ_x(t, x, s)\}, \quad (13)$$

In addition, the terminal conditions must be met as follows: $J(T, x, s) = v_T(x, s)$,

$$\lim_{s \rightarrow \infty} J_{ss}(t, x, s) \rightarrow 0, \quad \text{and} \quad \lim_{s \rightarrow 0^+} J_{ss}(t, x, s) \rightarrow 0.$$

Conversely, if a function $\tilde{J}(t, x, s)$ satisfies the preceding conditions, then $\tilde{J}(t, x, s) = J(t, x, s)$ for any $0 \leq t \leq T$, $0 \leq x \leq M$ and $s \geq 0$.

As with the peak-load storage model, the base-load facility enjoys a switching curve of the optimal injection and withdrawal decisions. The similarity enables us to omit the proof of the following theorem.

Theorem 6 *Suppose that for fixed s , $v_T(x, s)$ is a concave function of x for $x_0 \leq x \leq M$. The optimal value function $J(t, x, s)$ is concave in x . If $t < \tau$, then the optimal inject-up-to*

level can be defined as $I(t, s)$ in (10); if $t \geq \tau$, then the optimal withdrawal-down-to level can be defined as $W(t, s)$ in (11).

Other properties derived from the peak-load model can also be extended to the base-load storage model.

5 Price Forecast and Optimal Trading Rules

This section studies the impact of the price prediction on the optimal trading strategy and identifies a couple of opposite optimal trading rules in connection with different market forecast. These rules can be interpreted by traders to catch potential price moves. The optimal trading rules are established in a common faith in the future price moves. That is, a higher gas spot price projects a higher spot price in the future. This monotone prediction of the future spot price is mathematically described by ensuring assumption.

(A1) The spot price process $\{S(t) : t \geq 0\}$ is called to be *stochastically increasing* if for any $s_1 \geq s_2$ and any $s \geq 0$ and $t > t'$

$$P[S(t) \geq s | S(t') = s_1] \geq P[S(t) \geq s | S(t') = s_2].$$

By (A2), any stochastically increasing spot price process $\{S(t) : t \geq 0\}$ satisfies $E[f(S(t)) | S(t') = s_1] \geq E[f(S(t)) | S(t') = s_2]$ for $t > t'$, $s_1 > s_2$ and any increasing function f (Ross, 1996).

It is true for a continuous Markov process to be stochastically increasing. Because of jumps, however, it may not be sure for a general jump-diffusion process to be stochastically increasing.

Lemma 2 *The spot prices given by (2) and (3) are stochastically increasing.*

Proof. We first show that the mean-reverting process defined by (3) is stochastically increasing. In fact, $S(t)$ can be written to be the sum of two stochastic processes as follows: given $S(0) = s$,

$$S(t) = S_1(t) + S_2(t)$$

where $S_1(t)$ and $S_2(t)$ are given by

$$dS_1(t) = (\kappa(\theta - S_t) - \lambda\varpi)dt + \sigma_t dz_t, S_1(0) = s$$

and

$$dS_2(t) = (\xi_t - 1)dN(t), S_2(0) = 0,$$

respectively. Clearly, $S_1(t)$ is stochastically increasing and $S_2(t)$ is independent of s . For any increasing function f and real number y , let $\tilde{f}(y) = E[f(y + S_2(t)) | S_2(0) = 0]$ is increasing in y . Then, in view of

$$\begin{aligned} E[f(S(t)) | S(0) = 0] &= E[f(S_1(t) + S_2(t)) | S_1(0) = s, S_2(0) = 0] \\ &= E[E[f(S_1(t) + S_2(t)) | S_2(0) = s] | S_1(0) = s] = E[\tilde{f}(S_1(t)) | S_1(0) = s], \end{aligned}$$

$E[f(S(t)) | S(0) = s]$ is increasing in s . This deduces that $S(t)$ is stochastically increasing.

When $S(t)$ is a Geometric mean-reverting process given by (2), let $Y(t) = \ln S(t)$ be the logarithm of $S(t)$. By Itô lemma, $Y(t)$ is a mean-reversion process that is characterized by the following stochastic differential equation:

$$dY(t) = (\kappa(\theta - 1/2\sigma_t^2 - Y(t)) - \lambda\varpi)dt + \sigma_t dz_t + \ln \xi_t dN_t.$$

Thus, by application of the preceding analysis, $Y(t)$ is stochastically increasing. Since $\exp(x)$ is an increasing function of x , $S(t) = \exp(Y(t))$ is stochastically increasing as well. \square

Lemma 2 illustrates (A1) is a minor assumption and has been shared by many prevalent gas price processes. In the context of (A1), trading strategies exhibit certain commonly observable patterns or rules of thumb, some of which are quantitatively manifested by next theorem to be scientific rules.

Theorem 7 *Suppose that (A1) holds and $h(x, s) = \tilde{h}(x)$ is a convex function of x . Then, for any $t \in [0, T]$, the following assertions hold.*

1. $J(t, x, s)$ is supermodular in (x, s) .
2. Suppose that $\rho_i = \rho_w = 0$, $E[e^{-r\delta} S(\delta) | S(0) = s] - s$ is decreasing in s for any $\delta > 0$, and $\nu(x, s) - sx$ is submodular in (x, s) . Then,
 - (a) $J_x(t, x, s) - s$ is decreasing in s ,
 - (b) for fixed x $W(t, s)$ and $I(t, s)$ are decreasing in s ,
 - (c) for any x , $w(t, x, s)$ is increasing in s and $a(t, x, s)$ is decreasing in s .

Proof. See Appendix. \square

The first assertion of Theorem 7 says that optimal expected profit function is supermodular in spot price and gas inventory level. In other words, the marginal profit of an increase in

gas load increases in gas spot price, or equivalently, the marginal profit of an increase in gas price increases gas load. Therefore, an increase in gas spot price induces to inject gas or a decrease in gas spot causes to withdraw gas from underground. This illustrates that gas price and storage gas load are *economically complement*.

The assertions 2 and 3 erect a couple of optimal trading rules that state how the marketer should make current trading decision in response to the signals of the future price curve. These rules reflect insights similar to those of Ross and Zhu (2006) in valuing a swing contract. Assumption $\rho = \varrho = 1$ that results from the removal of the fuel charge is minor, just for ease of technical simplicity. Note that term $E[e^{-r\delta}S(\delta)|S(0) = s] - s$ is the present value of net profit received by purchasing one unit of natural gas at spot price s , storing it in the storage, and selling it in δ units of time later. Thus, assertion 2 holds assuming that the net profit of injection is positive. An appealing price process that induces this hypothesis for assertion 2 to stand is the mean-reverting price process defined in (3) (See Example 2 of Ross and Zhu [2006]). In addition, Geometric Brown Motion that is characterized by $dS(t)/S(t) = \mu_t dt + \sigma_t dz_t$ with the drift rate $\mu_t \leq r$ (See Example 3 of Ross and Zhu [2006]) also satisfies this hypothesis. Assertion 2 says that under this hypothesis, the marginal benefit of storing gas in the facility, $J_x(t, x, s) - s$, decreases in the spot price, yielding a decrease in both optimal withdrawal-down-to and inject-up-to levels. It is straightforward from assertion 2 that an increase in the spot price induces the marketer either to inject less or to withdraw more. The trading rule implied by assertion 2 states that when the perspective of a speculation is not rewarding, selling more gas is wise, or equivalently, buying gas should be deliberated, so as to keep gas load in the facility as lower as possible. This instructs to *sell high and buy low*, as a common rule in business practice.

5.1 Pressure of the Contractual Duration

This section concentrates on the effect of elapsed time changes on the optimal policy. To study the impact of time on the optimal policy, we first show the super-martingale properties of the opportunity cost.

Theorem 8 *When implementing the optimal operating strategy, the present value of dynamic marginal storage profit $e^{-rt} J_x(t, X_t, S_t)$ is a super-martingale, i.e., for any time interval $\delta \in (0, T - t)$,*

$$E[e^{-r(t+\delta)} J_x(t + \delta, X_{t+\delta}, S_{t+\delta}) | S_t = s, X_t = x] \leq e^{-rt} J_x(t, x, s). \quad (14)$$

In addition, if $E[e^{-r\delta}S_{t+\delta}|S_t = s] \geq S_t$, then

$$E\left[e^{-r\delta}\left((1 - \rho_w)S_{t+\delta} - c_w - J_x(t + \delta, X_{t+\delta}, S_{t+\delta})\right)\middle|S_t = s, X_t = x\right] \geq (1 - \rho_w)s - c_w - J_x(t, x, s).$$

In particular, if $c = 0$, we have

$$E\left[e^{-r\delta}\left(J_x(t + \delta, X_{t+\delta}, S_{t+\delta}) - (1 + \rho_i)S_{t+\delta}\right)\middle|S_t = s, X_t = x\right] \leq J_x(t, x, s) - (1 + \rho_i)s$$

Proof. See Appendix. □

The first assertion fits the intuition that the present value of marginal gas load of storage is diminishing over time on expectation. An illustration of this property is that, as time goes by, the remaining time to trade the gas with the storage facility becomes less and thus the opportunity/shadow price of the remaining gas in storage becomes less. The second assertion states that the present value of net benefit from the withdrawal operation increases on expectation with time past. On the contrary, the third assertion states that the present value of net benefit from the injection operation decreases on expectation with time past. This indicates that the pressure of the time passing forces the marketer to become more willing to withdraw than to inject. The condition $E[e^{-r\delta}S_{t+\delta}|S_t = s] \geq S_t$ is not very restrictive. According to the theory of competitive equilibrium of storable commodities (Williams and Wright 1990), if the aggregate industry-wide liquid inventory is positive, the market will reach equilibrium only when the spot prices satisfies $e^{-r\delta}E[S_{t+1}|S_t = s] = s + h'\delta$, where h' is the industrial average unit holding cost per unit of time. Furthermore, $c = 0$ represents that there is no variable injection cost other than the injection gas loss.

The following theorem investigates the monotonicity of the optimal policy over time.

Theorem 9 *Suppose $h(x, s) = 0$ and $v(x, s) = 0$. $J(t, x, s)$ is increasing in x for any (t, s) , $J(t, x, s)$ is decreasing in t for any (x, s) . Moreover, $J_x(t, x, s)$ is decreasing in t . Hence, for any x , $I(t, s)$ and $W(t, s)$ are decreasing in $t \in (0, T)$. Moreover, for all (x, s) , $w(t, x, s)$ is increasing in t and $a(t, x, s)$ is decreasing in t .*

Proof. The proof is in the Appendix. □

The assumption $h(x, s) = 0$ means that there is no holding cost, or, that the marketer does not need to pay the additional fee for the storage owner other than the injection/withdrawal/fuel charges. The assumption $v(T, x, s) = 0$ infers that the marketer will loss all the remaining inventory at the end of contract duration, which indicates that the inventory in storage under the storage contract is perishable.

Theorem 9 states that as time elapses, the opportunities cost will decrease and thus the optimal injection-up-to and withdrawal-down-to level are decreasing. The implication of the result is that, the less the remaining time to trade, the marketer should inject less and withdrawal more.

6 Firm Contract and Investment Decision

Based on the analysis of valuing a contract of the storage, this section further explores a few related strategic decisions to the storage valuation.

6.1 Valuing a Firm Storage Service Contract

A *firm storage contract* (FSS) gains a fractional capacity of the storage with guaranteed service over a period. The storage facility can also be operated by a storage service provider who designs and provides various storage services to marketers. There are two main types of prevalent storage service: *firm storage service* (FSS) and *interruptible service* (ISS). Firm storage service provides a customer with guaranteed space for gas within a storage facility (contracted volume) and a guaranteed rate at which the gas can be removed from the storage facilities (firm withdrawal). Interruptible storage service provides a customer with a contracted volume, but withdrawal from the facility is provided on a best effort basis. In the following, we deploy the structural properties of the storage model that we have developed to value the FSS contracts only, and then discuss the marketers' contracting strategies for favorable storage service contracts.

To evaluate an FSS contract, we must be aware of the specifications of a typical contract. Basically, an FSS contract specifies the requested storage services, such as *maximum daily injection quantity* (MDIQ), a *maximum daily withdrawal quantity* (MDWQ), a *maximum storage quantity* (MSD), and the respective service rates including a *storage capacity charge or reserve fee* (\$/MMbtu/Month), *injection/withdrawal charges* (\$/MMbtu), and a *fuel charge* (%). The excess injection/withdrawal and over-holding capacity charges can be much higher than contracted service rates. The minimum contract duration is one year unless otherwise specified in the service agreement.

We consider a generic FSS contract that specifies the maximum injection rate α with injection fee c , maximum withdrawal rate β , with withdrawal fee g and maximum reserved storage capacity M with reservation fee p . No excess injection/withdrawal and over-holding are al-

lowed. It is common in the gas storage industry to assume that $\alpha \geq \beta$, as competence in withdrawal periods is more intensive than that in injection periods. The operating charges and reservation fees postulated in the contract are scheduled to be paid in a sequence of fixed points in time prior to T . The present value of total reservation fees can be calculated by $\Psi(p, M) = pM \frac{1-e^{-rT}}{r}$. There is no holding cost for the marketer under the contract, i.e., $h(\cdot) \equiv 0$. Clearly, the FSS contract is equivalent to a special case of the whole-capacity contract that we presented in Section 4.1. Suppose that there is no gas inventory for the marketer in storage in the commerce of the lease. Let $\mathcal{C} = (\alpha, \beta, M, c, g, p)$ denote an FSS contract. Under \mathcal{C} , we denote by $J(t, x, s; \mathcal{C})$ the optimal value function, where (x, s) is the volume of gas and gas spot price at time t . The maximum value of this FSS contract is represented by $v(\mathcal{C}) = J(0, 0, s_0; \mathcal{C})$, where s_0 is the initial spot price. Thus, the net value of the FSS contract can be represented by

$$\Pi(\mathcal{C}) = v(\mathcal{C}) - \Psi(p, M).$$

The aim of optimizing contracting strategies, for fixed cost rates, is to seek contract parameters (α, β, M) to maximize $\Pi(\mathcal{C})$. Mathematically, the aim is written as

$$\max_{M, \alpha, \beta} \Pi(\mathcal{C}) = \max_{M, \alpha, \beta} \Pi(\alpha, \beta, M, c, g, p).$$

The following theorem addresses the effect of changing the operational constraints of an FSS contract.

Theorem 10 *Regarding operational characteristics as decisive variables, the following hold.*

1. *At a time t , $J(t, x, s; \mathcal{C})$ is concave in (x, α, β, M) and supermodular in $(x, -\alpha)$, (x, β) and (x, M) , respectively .*
2. *Optimal thresholds $I(t, s; \mathcal{C})$ and $W(t, s; \mathcal{C})$ are increasing in M and β , respectively, and decreasing in α .*

Under \mathcal{C} , Part 1 asserts that the marginal value of storage with the current gas load x , $J_x(t, x, s; \mathcal{C})$, is increasing in the contractual storage capacity, injection and withdrawal rate caps, which represent the flexibility of a contract. By virtue of Part 2, the higher the contractual storage capacity and deliverability, the more gas should be stored. Part 2 demonstrates that the flexibility of a contract enables the marketer to store more gas, and that increase can create more chances for speculation. Part 3 demonstrates that the marketer can benefit from increasing flexibility when the reservation and operating charges are fixed. These properties

are economically sound and reveal that the value of an FSS contract is the value of flexibility that the contract provides.

The following theorem will analyze the impact of the operational charges on delivery capability of an FSS contract.

Theorem 11 *After entering an FSS contract \mathcal{C} , $J(t, x, s; \mathcal{C})$ is submodular in (α, c) and (β, g) , respectively.*

Theorem 11 shows that the benefit from the improvement on the quality of a service or the flexibility or capacity of a given contract decreases in service charges. Thus, a better or more flexible service is not always worth seeking and it is possible to find out most favorable service menu from service provider.

The following theorem focuses on the determination of optimal capacity of an FSS contract, which is a consequence of Theorem 11.

Theorem 12 *The following hold.*

1. $\Pi(\mathcal{C})$ is increasing and concave in α and β , and concave in M . In addition, it is submodular in (α, c) , (β, g) , and (M, p) , respectively.
2. For fixed (α, β, c, g) , if there is no limit on the contractual capacity, then the optimal capacity of an FSS contract, $M^*(p)$, is determined by the first-order condition:

$$\frac{\partial \Pi(M^*(p), \alpha, \beta, p, c, g)}{\partial M} = 0.$$

In addition, $M^(p)$ is decreasing in p .*

According to Part 1 of the preceding theorem, the marginal value of contractual flexibility is reduced by the increases in the operating or reservation charges. For fixed operational capability and charges, as well as reservation fee, the optimal reservation capacity can be determined by the first-order condition. However, the capacity reservation fee affects on the optimal contracted capacity negatively. This conforms with our intuition.

In addition, for fixed injection and withdrawal charges, the marketer can reap a higher profit by gaining larger flexibility that is represented by contractual injection and/or withdrawal rate limits. In practice, the storage service provider usually publicizes the supply curves that plot the injection/withdrawal rate limits against capacity reservation charges, which are ordinarily

increasing functions $c(\alpha)$, $g(\beta)$, and $p(M)$, respectively. For fixed supply curves $c(\alpha)$, $g(\beta)$, and $p(M)$, the optimal contracting strategies of the marketer can be formulated into the following optimization problem:

$$\max_{\alpha, \beta, M} \Pi(M, \alpha, \beta, p(M), c(\alpha), g(\beta)).$$

The most favorable contracting flexibility must result from solving the first-order conditions of $\Pi(M, \alpha, \beta, p(M), c(\alpha), g(\beta))$ with respect to (M, α, β) .

6.2 Technology Deployment and Financial Investment

We consider the rent contract of a storage facility with capacity M units and effective duration T days. By the end of the contract, the marketer will pay (receive) from the storage owner the market value of the gas in the gas volume spread over the beginning and end of the contract. More accurately, any excess or deficit in the volume should be settled financially in the spot market at T . For simplicity, we assume that the initial volume is full or M , and then the terminal revenue function that must reflect the redemption of a deficit is equal to $v_T(x, s) = s(M - x)$, where x is the residential volume and s is the spot price at T . The rent contract is assessed by the maximum expected discounted trading profits cumulated less total cost over the duration of the contract. In this regard, the optimization analysis and the structural properties play a critically important role.

Suppose that the rental will be paid in a sequence of points of time prior to T . The present value of these rentals to the marketer is a lump-sum cost. Suppose that the initial volume of the storage is M . We then denote this lump-sum cost to be $\Psi(M)$, which depends on the initial spot price s_0 . The optimal value function of running the storage over T days is, in line with prevalent notation, $v(M) = J(0, M, s_0)$. The expected net profit of the rent contract is given by

$$\Pi(M) = v(M) - \Psi(M).$$

7 Numerical Analysis

7.1 Valuing a Natural Gas Storage Facility

This example is based on a sample natural gas storage facility (provided by Dr. Kevin G. Kindall, Commercial Division, ConocoPhillips) documented in Kjaer and Ronn (2006).

(1) Capacity: $M = 1\text{Bcf}$.

(2) Maximum injection/withdrawal rates are stepwise functions (in the industry, they are called “ratchets”) are given in Table 1.

(3) Injection cost: $\$0.0218/\text{mmBtu}$, withdrawal cost: $\$0.0195/\text{mmBtu}$. There is 3.59% injection fuel loss and no withdrawal fuel cost.

x/M	$\alpha(x)(\text{Btu/month})$	x/M	$\alpha(x)(\text{Btu/month})$
0 – 10%	$0.167M$	0 – 10%	$0.250M$
50 – 100%	$0.140M$	10 – 16%	$0.333M$
		16 – 30%	$0.375M$
		30 – 35%	$0.475M$
		35 – 100%	$0.500M$

Table 1: Injection/Withdrawal Rates (Ratchets)

We refer to Thompson et al. (2003) for a calibrated gas price process with jumps:

$$dS_t = 0.25(2.5 - S_t)dt + 0.2S_t dW_t + (\xi - S_t)dQ.$$

where Q is a Poisson process with intensity 2 and $\xi \in N(6, 4)$. The risk-free interest rate is 10% per year. The terminal value function $\nu(x, s) = 0$.

Then, the value function satisfies the Bellman equation (8) where

$$\begin{aligned} \tilde{\mathcal{L}}J &= J_t(t, x, s) + 0.5 \cdot 0.04s^2 J_{ss}(t, x, s) + 0.25 \cdot (2.5 - s)J_s(t, x, s) - rJ(t, x, s) \\ &+ 2 \int_0^\infty [J(t, x, \xi) - J(t, x, s)] \frac{1}{\sqrt{8\pi}} e^{-\frac{(\xi-s)^2}{8}} d\xi. \end{aligned}$$

We solve this problem with Clark-Nicolson finite difference numerical scheme. The numerical results are shown in Figures and .

Figure (2) shows the storage value surface which is a function of initial working gas level in storage and spot price. It reaches the highest value as gas price is high and the storage is full. In contrast, it reaches the lowest value if the storage is empty and as gas price is high. Given a spot price, the value function is increasing in inventory level. This is because there is no holding cost for gas inventory in storage, thus, at any time, the higher the inventory the higher the storage value.

Figure (3)-(6) demonstrate the optimal control strategy surface, which is represented by the net-withdrawal rate (=withdrwal rate-injection rate), for operating the gas storage facility with 3/4 year remaining in the contract. There are three control regimes. The positive region

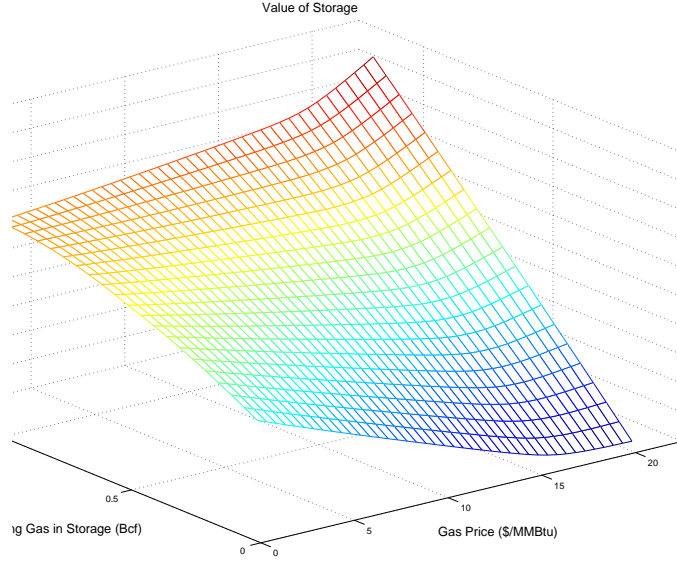


Figure 2: Value of Storage

corresponds to withdrawal (with maximum withdrawal rate). The negative region corresponds to injection. The zero region between the negative and positive regions corresponds to *no transaction*. When the inventory level in storage is low and gas price is low, it is optimal to inject (negative region). When the gas price is high, it is optimal to withdrawal (positive region). The negative (injection) region becomes deeper for lower gas inventory level in storage. This is because the injection rate is decreasing in inventory level. Since the withdrawal rate is increasing in inventory level in storage, in positive (withdrawal) region, the higher the gas inventory in storage is the higher the withdrawal rate will be. In general, for any given price, the net withdrawal rate is increasing in inventory level since . This is because it becomes more expensive to inject into an storage with more gas. For any given inventory level, the net withdrawal rate is increasing in spot price. This reflects the *buying low and selling high* characteristics of the spot market trading.

Moreover, comparing the withdrawal rate surface at various time, we can observe that the positive region becomes thicker and the negative and zero regions become thinner. This is because, when the remaining time becomes less, the opportunity to extract gas from storage becomes less, so it is better to withdrawal more and inject less.

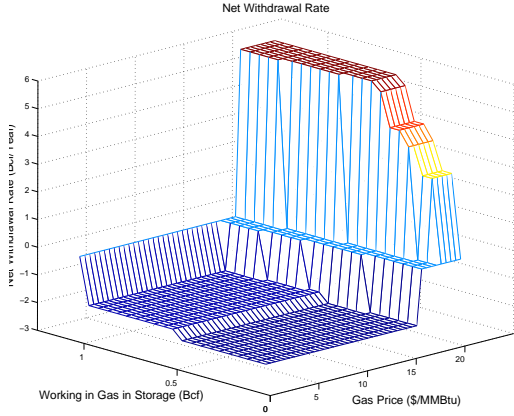


Figure 3: Net withdrawal rate at $t=0$

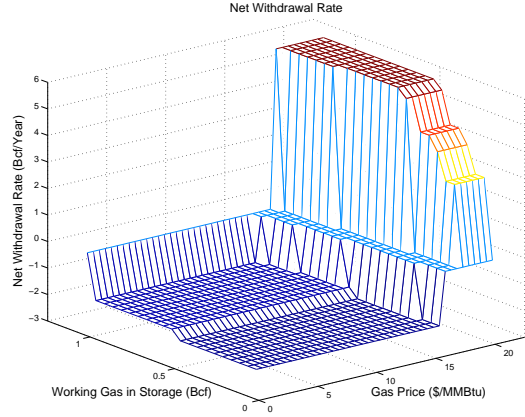


Figure 4: Net withdrawal rate at $t=\frac{1}{4}$

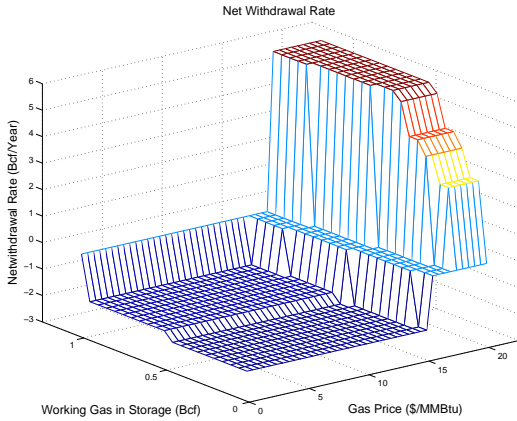


Figure 5: Net withdrawal rate at $t=\frac{1}{2}$

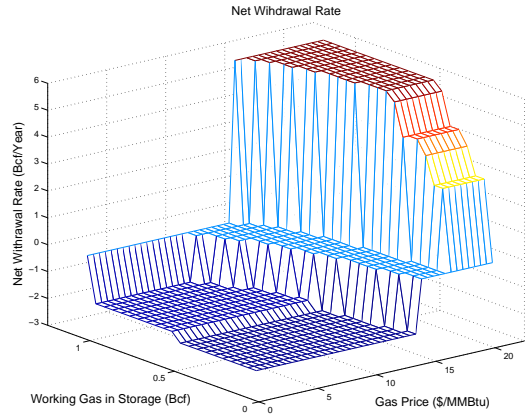


Figure 6: Net withdrawal rate at $t=\frac{3}{4}$

7.2 Valuing a Firm Storage Contract

Consider a firm storage contract with a total capacity of 8Bcf rented out for one year ($T=1$). The price process is calibrated from the data provided by de Jong and Walet (2003),

$$d \log(S_t) = \kappa \cdot (\log(\eta) - \log S_t) dt + \sigma dB_t,$$

where $\kappa = 17.1$, $\eta = 3$ and $\sigma = 1.33$. The initial inventory is 4Bcf and the terminal value function is

$$\nu(x, s) = -2 \cdot s \cdot \max\{4 - x, 0\},$$

which infers that the contract holder has to pay double price for each unit of the final inventory less than the initial inventory, 4Bcf, and gets no refund for any excess (over 4Bcf). The

maximum injection rate (yearly units) $\alpha(x) \equiv 0.06 \cdot 365$ (Bcf/year) and the maximum withdrawal rate (yearly units) $\beta(x) \equiv 0.25 \cdot 365$ (Bcf/year). The maximum injection/withdrawal rates are assumed to be independent of inventory levels. The risk-free interest rate is $r = 0.06$. There are 1% injection and withdrawal fuel costs, i.e., $\rho_i = \rho_w = 0.01$. The variable injection/withdrawal costs for injection/withdrawal is $c_i = \$0.02/\text{MMBtu}$ and $c_w = \$0.01/\text{MMBtu}$. The yearly holding cost rate is $h = \$0.1/\text{MMBtu}$. If the initial spot price is $\$3/\text{MMBtu}$, the value of the storage is $V(0, 4, 3) = \$8.823 \cdot 10^6$. In the following, we will vary the price dynamics (by changing volatility and mean-reverting rate) and the operational characteristics (injection/withdrawal rate and storage capacity) so as to exploit the relationships between storage value and the market and operational characteristics.

Sensitivity of Price Dynamics

As we have introduced, the intrinsic value of storage can be captured with the spread of the forward curve, while the extrinsic value of storage relies on the volatility of gas spot price which can only be fully exploited by optimizing the injection/withdrawal operations. To exploit the relationship between spot price (yearly) volatility and value of the storage contract, we calculate the storage value with respect to different volatility levels, which is demonstrated in Table 2. We can see in Table 2 that the higher the volatility of the spot price, the higher the value of the storage. As the extrinsic value of the storage comes from the volatility of the spot price, so it is nature to expect that a higher volatility creates more value for a storage. This intuition is justified by this example. Another interesting observation is that the storage value increases almost linearly in volatility and the slope is not very sensitive to the spot price as shown in Figure 7.

The Table 2 and figure 8 show that the value of storage is first increasing and then decreasing in mean-reverting rate. Comparing the effects of volatility and mean-reverting rate on storage value, we can see that the storage value is more sensitive to the volatility in general. This further fixes into our intuition that the main driver of the extrinsic value of the storage is volatility.

Sensitivity of Operational Characteristics

Besides the price dynamics, the operational characteristics also have remarkable impacts on the value of storage. Table 3 below examines the effects of changing the capacities (injection rate, withdrawal rate and working gas capacity) on the value of storage.

Value of storage facility $V(0, 4, S(0))$ (10^6 \$)							
σ	Initial spot price (\$/MMBtu)						
	1	2	3	4	5	6	7
0.5	3.0976	1.8009	1.1197	2.4051	3.8582	4.8806	5.1335
1	7.2839	6.0326	5.4582	6.1606	7.8319	10.1359	12.9728
1.5	12.2568	11.2124	10.7807	11.2012	12.495	14.5216	17.1743
2	18.0543	17.2044	16.9011	17.2311	18.22	19.98	22.4234
2.5	24.7268	24.0654	23.8823	24.1917	24.9758	26.4977	28.7353
κ	Initial spot price (\$/MMBtu)						
	1	2	3	4	5	6	7
1	4.7038	4.9239	4.8762	4.945	5.2148	5.8758	7.2044
5	10.5621	9.3526	8.9649	9.5872	11.0574	13.7229	17.4647
10	11.4469	10.1095	9.5884	10.1226	11.6696	14.1657	17.4717
15	10.8296	9.6464	9.1499	9.6463	11.1033	13.3322	16.2101
20	9.9604	8.9389	8.4917	8.9769	10.3144	12.2982	14.822

Table 2: Sensitivity analysis of price dynamics

Obviously, we can observe that the value of storage is increasing and concave in the injection rate, withdrawal rate and working gas capacity, respectively. The higher the operational capacities, the more flexibilities the storage have and thus the more extrinsic value it can create, and the increments will become less as the capacities become larger, as we have gained analytically in section 6.1.

Comparing the sensitivities of the injection rate (through calculating the slope of the value function with respect to the corresponding capacity parameters), withdrawal rate and the working capacities to the value of storage, we find that the injection rate has much greater impact than the other two capacity factors. A possible illustration is that, compared with withdrawal rate and working gas capacity, the injection rate is the bottleneck of the facility (contract), so that the changing of injection rate has larger marginal effect than the other factors. This enlightens us that we should pay more attention on the determination of injection rates as we enter a storage contract if we can identify that the injection rate is the bottleneck factor of the contract.

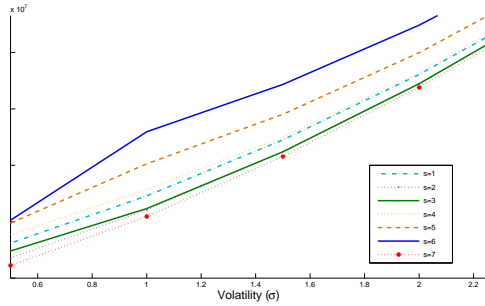


Figure 7: Sensitivity of volatility

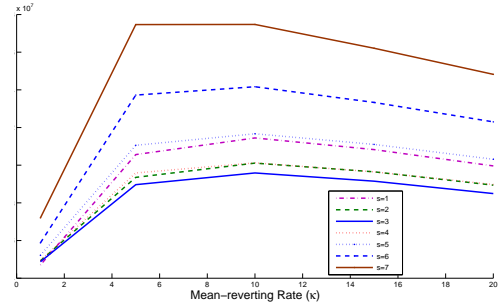


Figure 8: Sensitivity of mean-reverting rate

8 Concluding Remark

We have analyzed the valuation of a storage rent contract for either a peak- or a base-load storage facility via real options approach. The valuation is based on the estimated maximum revenue of the storage during the contractual duration in terms of predictable spot price evolution in the future. The liquid and accessible spot market makes trading on a daily basis more profitable than trading with a forwards or the mixture of spot and forwards strategies. This occurs because the probability of some forwards contracts ending out-of-the-money is not insignificant, and forwards curves are unpredictable. Thus, we calibrate a continuous-time Markov decision process framework in optimizing injection and withdrawal decisions in terms of day-ahead forward predictability. We find that the facility's day-to-day loading and releasing can be optimized by a switching curve policy for each day's spot gas price. In particular, with the current gas load level known, the optimal policy advises withdrawal or injection above and below certain price levels. Most importantly, the rent contract is valued by resorting to the concavity of the optimal revenue function, and is effectively realized with the aid of a numerical algorithm. This algorithm is developed by modifying the Clark-Nicolson numerical finite difference algorithm of pricing financial options. Our contribution includes the retention of the optimal switching curve policy for daily gas trading in a more general liquid spot market and the development of an effective algorithm in response to real-time schedules.

The issue of storage valuation and optimal operation is not limited to gas market. Storage also play important roles in other storable commodity markets such as oil markets (through oil storages/tanks), hydropower electricity markets (through pumped-storage hydropower stations), and agriculture markets (through warehouses). The analysis in our paper are applicable to those markets as well (by incorporating features of corresponding markets such as operational constraints and market dynamics) as long as there is spot markets for the underlying

Value of Storage ($V(0, 4, s)$) (10^6 \$)									
$\alpha(x)$	$\beta(x)$	M	Initial Spot Price (\$/MMBtu)						
			1	2	3	4	5	6	7
0.01	0.25	8	2.5423	2.3357	2.2564	2.4056	3.1484	4.8093	7.2195
0.05	0.25	8	9.4928	8.5354	8.1254	8.5653	9.9344	12.0395	14.7544
0.1	0.25	8	13.1912	11.5808	10.8936	11.5381	13.0373	15.1968	17.9282
0.15	0.25	8	15.1745	13.1512	12.274	13.0264	14.5814	16.7584	19.4908
0.2	0.25	8	16.4279	14.1359	13.1374	13.9542	15.5374	17.724	20.4561
$\alpha(x)$	$\beta(x)$	M	Initial Spot Price (\$/MMBtu)						
			1	2	3	4	5	6	7
0.06	0.06	8	6.3395	5.3021	4.988	5.4565	6.0913	6.9094	7.8978
0.06	0.1	8	7.9626	6.8845	6.4894	7.0029	7.8918	9.0997	10.5956
0.06	0.15	8	9.1611	8.0631	7.6244	8.1445	9.2624	10.85	12.8533
0.06	0.2	8	9.9335	8.8256	8.3655	8.8745	10.152	12.035	14.4373
0.06	0.25	8	10.4716	9.3591	8.8823	9.3736	10.7798	12.9017	15.6221
$\alpha(x)$	$\beta(x)$	M	Initial Spot Price (\$/MMBtu)						
			1	2	3	4	5	6	7
0.06	0.25	5	7.1643	6.8641	6.9375	7.9434	9.5578	11.7898	14.5758
0.06	0.25	6	8.699	7.9732	7.78	8.572	10.1022	12.2908	15.0512
0.06	0.25	7	9.7442	8.7767	8.4163	9.0376	10.4985	12.6499	15.388
0.06	0.25	8	10.4716	9.3591	8.8823	9.3736	10.7798	12.9017	15.6221
0.06	0.25	10	11.3535	10.0849	9.4718	9.7825	11.1075	13.1852	15.8795

Table 3: Sensitivity analysis of operational characteristics

commodities that allow liquid spot trading. We refer to Ludkovsky and Carmona (2005) for a further discussion of the applications of this model in hydro-electric and emission trading.

However, analogy to the literature on storage asset valuation, the purpose of this model is on pricing. In terms of risk management, the gas storage facility is used to hedge the risk instead of arbitrage or speculation. A challenge problem arises here: How to incorporate natural storage facility in the hedging strategies of energy service provider.

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