

Model Disagreement and Economic Outlook*

Daniel Andrei[†] Bruce Carlin[‡] Michael Hasler[§]

October 3, 2014

Abstract

We study the impact of model disagreement on the dynamics of asset prices and return volatility. In our framework, two investors have homogeneous preferences and equal access to information, but disagree about the length of the business cycle. Model disagreement induces agents to have different economic outlooks and generates trading activity and time-varying volatility. We show that model disagreement is the primary cause of persistent fluctuations in stock market volatility and that it induces a GARCH-like behavior of stock returns. Further, we show that volatility increases significantly with disagreement in bad economic times, whereas this relationship is weaker during good times. We test these theoretical predictions empirically and find statistically significant evidence for them.

Keywords: Asset Pricing, Learning, Disagreement, Economic Outlook, Volatility, GARCH

*We would like to thank Tony Berrada, Mike Chernov, Julien Cujean, Alexander David, Jerome Detemple (EFA discussant), Bernard Dumas, Barney Hartman-Glaser, Julien Hugonnier, Arvind Krishnamurthy, Ali Lazrak (NFA discussant), Francis Longstaff, Hanno Lustig, Monika Piazzesi, Nick Roussanov (Cavalcade discussant), Martin Schneider, Ken Singleton, Pascal St.-Amour, Wei Xiong, and Hongjun Yan for their useful advice. We would also like to acknowledge comments from conference and seminar participants at the SFI meeting in Gerzensee, the 4th International Forum on Long-Term Risks, UCLA Anderson, the 2014 Mathematical Finance Days, the 2014 SFS Finance Cavalcade, the 2014 WFC, the 2014 SITE Summer Workshop, the 2014 EFA meeting, and the 2014 NFA meeting. Financial support from the Swiss Finance Institute, NCCR FINRISK of the Swiss National Science Foundation, UCLA, and the University of Toronto is gratefully acknowledged.

[†]UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C420, Los Angeles, CA 90095, USA, daniel.andrei@anderson.ucla.edu, danielandrei.info

[‡]UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C413, Los Angeles, CA 90095, USA, bruce.carlin@anderson.ucla.edu, <http://www.anderson.ucla.edu/faculty/finance/faculty/carlin>

[§]University of Toronto, Rotman School of Management, 105 St. George Street, Toronto, ON, M5S 3E6, Canada, michael.hasler@rotman.utoronto.ca, www.rotman.utoronto.ca/Faculty/Hasler.aspx

1 Introduction

The field of finance is currently grappling with the fact that there are limits to applying the standard Bayesian paradigm to asset pricing. Specifically, in a standard Bayesian framework, beliefs are updated with a particular model in mind. However, as noted by Hansen and Sargent (2007), many economic models cannot be trusted completely, thereby introducing the notion of model uncertainty. Theoretically, though, as long as the potential set of models that all agents in an economy consider is the same ex ante, the Bayesian framework can still apply because agents can update their beliefs about which model explains the economy. However, if the agents consider different sets of models or they adhere to different paradigms, then disagreement will persist regarding which model is best to describe the world or predict the future (Acemoglu, Chernozhukov, and Yildiz, 2009). It is this notion of *model disagreement* that we focus on in this paper and characterize its effects on asset prices, return volatility, and trade in the market.

Empirically, model disagreement appears to be important. For example, in a recent paper by Carlin, Longstaff, and Matoba (2014), the authors study the effects of disagreement about prepayment speed forecasts in the mortgage-backed securities market on risk premia, volatility, and trading volume. Indeed, the prepayment models that traders use are often proprietary and differ from each other, while the inputs to these models are publicly observable (e.g., unemployment, interest rates, inflation). In that paper, the authors show that disagreement is associated with a positive risk premium and is the primary channel through which return volatility impacts trading volume.

In this paper, we analyze a continuous-time framework in which investors exhibit model disagreement and study how this affects the dynamics of asset prices. In our setup, two investors have homogenous preferences and equal access to information, but disagree about the length of the business cycle. Each investor knows that the expected dividend growth rate mean-reverts, but uses a different parameter that governs the rate at which this fundamental returns to its long-term mean. The disagreement is commonly known, but each agent adheres to his own model when deciding whether to trade.

Using disagreement about the length of the business cycle is natural and plausible. For example, Massa and Simonov (2005) show that forecasters strongly disagree on recession probabilities, which implies that they have different beliefs regarding the duration of recessionary and expansionary phases. The origin of this disagreement may arise from many sources. Indeed, there still remains much debate regarding the validity of long-run risk models (e.g., Beeler and Campbell 2012; Bansal, Kiku, and Yaron 2012). Additionally, in practice agents might use different time-series to estimate the mean-reversion parameter (e.g., use consumption versus production data). Likewise, their estimation methods may

differ (e.g., fitting the model to past analyst forecast data versus a moving-average of output growth versus performing maximum-likelihood Kalman filter estimation). Finally, as Yu (2012) documents, least-squares and maximum-likelihood estimators of the mean-reversion speed of a continuous-time process are significantly biased. Some investors might be aware of the existence of this bias and would adjust their estimation accordingly, whereas other investors might ignore it.¹

In our equilibrium model, two distinct quantities turn out to be important determinants of asset prices and trade in the market. The first is the *disagreement over fundamentals*, which is the instantaneous difference in beliefs about the expected growth rate in the economy. The second is the *difference in economic outlooks*, which affects expectations of future economic variables and takes into account how both agents will disagree over fundamentals in the future. Both quantities generate trading volume and excess volatility, in line with existing results (Harris and Raviv 1993, Dumas, Kurshev, and Uppal 2009). Our main contribution is to document a clear link between the persistence of disagreement arising in our model and the persistence of stock market volatility.² To identify this link, we disentangle the impact of disagreement from the impact generated by the other driving forces by decomposing stock return volatility. We show that, indeed, disagreement is the main driving force of persistent fluctuations in stock market volatility, whereas the level of the volatility is mainly driven by long-run risk, as the long-run risk literature (Bansal and Yaron, 2004) suggests.³

Our results help to explain three well-known characteristics about financial market volatility. First, volatility systematically exceeds that justified by fundamentals (Shiller, 1981; LeRoy and Porter, 1981). Indeed, we show that model disagreement amplifies volatility, over and above the usual effect of uncertainty. Second, volatility is time-varying (Schwert, 1989; Mele, 2008). This arises naturally out of our model because disagreement is mean-reverting. Last, but most important, volatility is persistent (Engle, 1982; Bollerslev, 1986; Nelson, 1991), occurring in clusters. This persistence (or predictability) has been described

¹This form of disagreement arises if agents are uncertain about the interpretation of public information, even after observing infinitely many signals (Acemoglu, Chernozhukov, and Yildiz, 2009). We further justify the assumption of different parameters in Appendix A.1 by performing a simulation exercise in which we let the agents estimate the mean-reversion parameter with different methods. We show that the difference between the estimated parameters is typically substantial, even though we perform 1,000 simulations of economies of length of 50 years at quarterly frequency.

²Persistent disagreement is consistent with empirical findings by Patton and Timmermann (2010) and Andrade, Crump, Eusepi, and Moench (2014).

³We define long-run risk here as the risk associated to a persistent expected dividend growth rate only. In Bansal and Yaron (2004) long-run risk captures the risk associated to a persistent expected dividend growth rate, a persistent dividend growth volatility, and a persistent expected dividend growth volatility. In contrast, we do not assume that any fundamental variable features stochastic volatility. Instead, stock return volatility becomes stochastic in equilibrium exactly because agents disagree about the magnitude of long-run risk. We thus argue that long-run risk *per se* is not a cause of fluctuations in volatility, whereas *disagreement about long-run risk* endogenously gives rise to such fluctuations.

extensively in the empirical literature, but there is a paucity of theoretical explanations.⁴ We show that model disagreement generates a new channel of persistence transmission from investors beliefs to stock market volatility and we fit a GARCH model on simulated stock returns to show that volatility is indeed persistent. Our paper proposes therefore a theoretical foundation for the GARCH-like behavior of stock returns.

To build on this, we empirically test two new predictions of our model. First, using the volatility of the S&P 500 as a proxy for volatility and the dispersion of analyst forecasts of the one-quarter-ahead U.S. GDP growth rate as a proxy for disagreement, we find a significant and positive correlation between the persistence of disagreement and the persistence of volatility, confirming our main theoretical implication. Second, our model predicts that the relationship between disagreement and volatility is stronger in bad economic times (i.e., when agents forecast low expected growth rates). Using the same data, we run predictive regressions of future volatility on lagged disagreement and find that volatility increases significantly with disagreement only during bad economic times, consistent with our theoretical results.

Finally, we conclude the paper with a survival analysis. Indeed, in any model with heterogeneous agents, whether all types survive in the long-run is a reasonable concern. To address this, we perform simulations and show that all agents in our economy with model disagreement survive for long periods of time, consistent with previous findings in the literature (Yan, 2008). Based on this, we posit that model disagreement can have long-lasting effects on asset prices without eliminating any players from the marketplace, which likely makes our analysis economically important.

Our approach contrasts with previous work and thus adds to the previous finance literature. As already mentioned, Hansen and Sargent (2007) studies model misspecification and model uncertainty, but does so for a single investor.⁵ In contrast, our study investigates the consequence of disagreement about models in an economy with different investors. We assume that investors disagree about the model governing the economy. Certainly, there are many other forms of disagreement; in particular, several papers feature a setting in which investors agree on the model governing the economy but disagree on the information that they receive (see, e.g., Scheinkman and Xiong 2003, Dumas, Kurshev, and Uppal 2009, or Xiong and Yan 2010). These models are able to generate excess volatility but they do not identify the cause of persistent fluctuations in volatility. In contrast with the rest of theoretical literature on disagreement, we propose—and find empirical support for—a foundation of the GARCH-like dynamics in volatility. Furthermore, we also document, both theoretically

⁴Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001), and McQueen and Vorkink (2004) provide preference-based foundations for volatility clustering.

⁵See also Uppal and Wang (2003), Maenhout (2004), Liu, Pan, and Wang (2005), and Drechsler (2013).

and empirically, an asymmetric relationship over the business cycle between disagreement and volatility.⁶

The remainder of the paper is organized as follows. Section 2 describes the model and its solution. Section 3 explores how model disagreement affects the dynamics of volatility. Section 4 addresses the survival of investors. Section 5 concludes. All derivations and computational details are in Appendix A.

2 Model Disagreement

Consider a pure exchange economy defined over a continuous time horizon $[0, \infty)$, in which a single consumption good serves as the numéraire. The underlying uncertainty of the economy is characterized by a 2-dimensional Brownian motion $W = \{(W_t^\delta, W_t^f) : t > 0\}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The aggregate endowment of consumption is assumed to be positive and to follow the process:

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dW_t^\delta \quad (1)$$

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f, \quad (2)$$

where W^δ and W^f are two independent Brownian motions under the physical (objective) probability measure \mathbb{P} . The expected consumption growth rate f , henceforth called *the fundamental*, is unobservable and mean-reverts to its long-term mean \bar{f} at the speed λ . The parameters σ_δ and σ_f are the volatilities of the consumption growth and of the fundamental.

There is a single risky asset (the stock), defined as the claim to the aggregate consumption stream over time. The total number of outstanding shares is unity. In addition, there is also a risk-free bond, available in zero-net supply.

The economy is populated by two agents, A and B . Each agent is initially endowed with equal shares of the stock and zero bonds, can invest in these two assets, and derives utility from consumption over his or her lifetime. Each agent chooses a consumption-trading policy to maximize his or her expected lifetime utility:

$$U_i = \mathbb{E}^i \left[\int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\alpha}}{1-\alpha} dt \right], \quad (3)$$

where $\rho > 0$ is the time discount rate, $\alpha > 0$ is the relative risk aversion coefficient, and

⁶The literature on differences in beliefs is large. See, among many others, Varian (1985), Harris and Raviv (1993), Detemple and Murthy (1994), Kandel and Pearson (1995), Zapatero (1998), Scheinkman and Xiong (2003), Li (2007), Cao and Ou-Yang (2009), David (2008), Xiong and Yan (2010), Chen, Joslin, and Tran (2010, 2012), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013), Buraschi and Whelan (2013), and Buraschi, Trojani, and Vedolin (2014).

c_{it} denotes the consumption of agent $i \in \{A, B\}$ at time t . The expectation in (3) depends on agent i 's perception of future economic conditions. Agents value consumption streams using the same preferences with identical risk aversion and time discount rate but, as we will describe below, have heterogeneous beliefs.

2.1 Learning and Disagreement

The agents commonly observe the process δ , but have incomplete information and heterogeneous beliefs about the dynamics of the fundamental f . Specifically, the agents agree that the fundamental mean-reverts but disagree on the value of the mean-reversion parameter λ . As such, they have different perceptions about the length of the business cycle.⁷

Agent A 's perception of the aggregate endowment and the fundamental is

$$\begin{aligned}\frac{d\delta_t}{\delta_t} &= f_{At}dt + \sigma_\delta dW_{At}^\delta \\ df_{At} &= \lambda_A (\bar{f} - f_{At}) dt + \sigma_f dW_{At}^f,\end{aligned}$$

where W_A^δ and W_A^f are two independent Brownian motions under agent A 's probability measure \mathbb{P}^A . On the other hand, agent B believes that

$$\begin{aligned}\frac{d\delta_t}{\delta_t} &= f_{Bt}dt + \sigma_\delta dW_{Bt}^\delta \\ df_{Bt} &= \lambda_B (\bar{f} - f_{Bt}) dt + \sigma_f dW_{Bt}^f,\end{aligned}$$

where W_B^δ and W_B^f are two independent Brownian motions under agent B 's probability measure \mathbb{P}^B . Both agents agree on the long-term mean of the fundamental \bar{f} and on the volatility of the fundamental σ_f .⁸

Neither agent uses the right parameter λ . Instead, the true parameter λ is assumed to lie somewhere in between the parameters perceived by the agents. As such, there are 3 probability measures: the objective probability measure \mathbb{P} and the two probability measures \mathbb{P}^A and \mathbb{P}^B as perceived by agents A and B .

The agents both observe the aggregate endowment process δ and use it to estimate

⁷Asset pricing implications of heterogenous models and parameters are provided in David (2008), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013), Buraschi and Whelan (2013), and Buraschi, Trojani, and Vedolin (2014) among others.

⁸We have considered extensions of the model where agents have heterogeneous parameters \bar{f} and σ_f , with similar results. The parameter bearing the main implications is the mean-reversion speed λ and thus we choose to focus on heterogeneity about it and to isolate our results from other sources of belief heterogeneity.

the fundamental f under their respective probability measures.⁹ Since they use different models, they have different estimates of f . Define \widehat{f}_A and \widehat{f}_B as each agent's estimate of the unobservable fundamental f :

$$\widehat{f}_{it} \equiv \mathbb{E}_t^i[f_{it}], \quad \text{for } i \in \{A, B\},$$

which are computed using standard Bayesian updating techniques. Learning is implemented via Kalman filtering and yields¹⁰

$$d\widehat{f}_{it} = \lambda_i \left(\bar{f} - \widehat{f}_{it} \right) dt + \frac{\gamma_i}{\sigma_\delta} d\widehat{W}_{it}^\delta, \quad \text{for } i \in \{A, B\},$$

where γ_i denotes the posterior variance perceived by agent i and \widehat{W}_i^δ represents the normalized innovation process of the dividend under agent i 's probability measure

$$d\widehat{W}_{it}^\delta = \frac{1}{\sigma_\delta} \left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{it} dt \right). \quad (4)$$

The process in Equation (4) has a simple interpretation. Agent i observes a realized growth of $d\delta_t/\delta_t$ and has an expected growth of $\widehat{f}_{it}dt$. The difference between the realized and the expected growth, normalized by the standard deviation σ_δ , represents the surprise or the innovation perceived by agent i .

The posterior variance γ_i (i.e., *Bayesian uncertainty*) reflects incomplete knowledge of the true expected growth rate. It is defined by¹¹

$$\gamma_i \equiv \text{Var}_t^i[f_{it}] = \sigma_\delta^2 \left(\sqrt{\lambda_i^2 + \frac{\sigma_f^2}{\sigma_\delta^2}} - \lambda_i \right) > 0, \quad \text{for } i \in \{A, B\}. \quad (5)$$

Equation (5) shows how γ_i depends on the initial parameters. The posterior variance increases with the volatility of the fundamental σ_f and with the volatility of the aggregate endowment σ_δ , and decreases with the mean-reversion parameter λ_i . Intuitively, if λ_i is small then agent i believes the process f to be persistent and thus the perceived uncertainty

⁹We assume that the only public information available is the history of the aggregate endowment process δ . The model can also accommodate public news informative about the fundamental, but here we chose not to obscure the model's implications and we abstract away from additional public news. The effects of heterogeneous beliefs about public news are well-understood (see, e.g., Scheinkman and Xiong (2003) or Dumas, Kurshev, and Uppal (2009) among others).

¹⁰See Theorem 12.7 in Liptser and Shiryaev (2001) and Appendix A.2 for computational details.

¹¹As in Scheinkman and Xiong (2003) or Dumas, Kurshev, and Uppal (2009), we assume that the posterior variance has already converged to a constant. The convergence arises because investors have Gaussian priors and all variables are normally distributed. This generates a deterministic path for the posterior variance and a quick convergence (at an exponential rate) to a steady-state value.

in the estimation is large. Since agents A and B use different mean-reversion parameters, it follows that their individual posterior variances are different, that is, one of the agents will perceive a more precise estimate of the expected growth rate. Therefore, one of the agents appears “overconfident” with respect to the other agent, although overconfidence here does not arise from misinterpretation of public signals as in [Scheinkman and Xiong \(2003\)](#) or [Dumas, Kurshev, and Uppal \(2009\)](#), but from different underlying models.

The innovation processes \widehat{W}_A^δ and \widehat{W}_B^δ are Brownian motions under \mathbb{P}^A and \mathbb{P}^B , respectively. They are such that agent i has the following system in mind

$$\frac{d\delta_t}{\delta_t} = \widehat{f}_{it}dt + \sigma_\delta d\widehat{W}_{it}^\delta \quad (6)$$

$$d\widehat{f}_{it} = \lambda_i(\bar{f} - \widehat{f}_{it})dt + \frac{\gamma_i}{\sigma_\delta} d\widehat{W}_{it}^\delta, \quad i \in \{A, B\}. \quad (7)$$

A few points are worth mentioning. First, although the economy is governed by two Brownian motions under the objective probability measure \mathbb{P} (as shown in (1)-(2)), there is only one Brownian motion under each agent’s probability measure \mathbb{P}^i . This arises because there is only one *observable* state variable, the aggregate endowment δ . Second, the instantaneous variance of the observable process δ is the same for both agents, which is not the case for the instantaneous variance of the filter \widehat{f}_i . Because of the “overconfidence” effect induced by different parameters λ , one of the agents will perceive a more volatile filter than the other.

Furthermore, agreeing to disagree implies that each agent knows how the other agent perceives the economy and that they are aware that their different perceptions will generate disagreement—*although they observe the same process* δ . This important feature (that the aggregate endowment process is observable and thus it should be the same for both agents) provides the link between the two probability measures \mathbb{P}^A and \mathbb{P}^B . Writing the aggregate consumption process (6) for both agents and restricting the dynamics to be equal provides a relationship between the innovation processes \widehat{W}_A^δ and \widehat{W}_B^δ (technically, a change of measure from \mathbb{P}^A to \mathbb{P}^B):

$$d\widehat{W}_{At}^\delta = d\widehat{W}_{Bt}^\delta + \frac{1}{\sigma_\delta} (\widehat{f}_{Bt} - \widehat{f}_{At}) dt. \quad (8)$$

Equation (8) shows how one can convert agent A ’s perception of the innovation process \widehat{W}_A^δ to agent B ’s perception \widehat{W}_B^δ . The change of measure consists of adding the drift term on the right hand side of (8). For example, suppose that agent A has an estimate of the expected growth rate of $\widehat{f}_{At} = 1\%$, whereas agent B ’s estimate is $\widehat{f}_{Bt} = 3\%$. Assume that the realized growth rate (observed by both agents) turns out to be $d\delta_t/\delta_t = 2\%$. It follows that agent B was optimistic and $d\widehat{W}_{Bt}^\delta = -0.01/\sigma_\delta$, whereas agent A was pessimistic and $d\widehat{W}_{At}^\delta = 0.01/\sigma_\delta$.

The extra drift term in Equation (8) comprises the difference between each agent’s estimates of the growth rate ($\widehat{f}_{Bt} - \widehat{f}_{At}$) or the *disagreement*, which we denote hereafter by \widehat{g}_t . We can now use this relationship to compute the dynamics of \widehat{g}_t , under one of the agent’s probability measure, say \mathbb{P}^B .

Proposition 1. (*Evolution of Disagreement*) *Under the probability measure \mathbb{P}^B , the dynamics of disagreement are given by*

$$d\widehat{g}_t = d\widehat{f}_{Bt} - d\widehat{f}_{At} = \left[(\lambda_A - \lambda_B)(\widehat{f}_{Bt} - \bar{f}) - \left(\frac{\gamma_A}{\sigma_\delta^2} + \lambda_A \right) \widehat{g}_t \right] dt + \frac{\gamma_B - \gamma_A}{\sigma_\delta} d\widehat{W}_{Bt}^\delta. \quad (9)$$

Proof. See Appendix A.3 □

Proposition 1 characterizes the dynamics of disagreement¹², which yields several properties that make it different from previous models of overconfidence that have been studied in the literature. First, if one of the agents believes in long-run risk, disagreement is persistent. Second, if agents have different degrees of precision in their estimates (which happens to be the case when they use different parameters λ), disagreement is stochastic. Third, because its long-term drift is stochastic, disagreement will never converge to a constant but will always be regenerated—even without a stochastic term.

To see this, observe that Equation (9) shows that disagreement is mean-reverting around a stochastic mean, driven by \widehat{f}_B . This arises because $\lambda_A \neq \lambda_B$. If agents adhered to the same models, disagreement would revert to zero, as in Scheinkman and Xiong (2003) and Dumas, Kurshev, and Uppal (2009). In contrast, in our setup, the mean is driven by \widehat{f}_B because the agents use different models. In addition, if one of the agents, say agent B , believes in long-run risk, disagreement becomes persistent because it mean reverts to a persistent \widehat{f}_B .¹³

To appreciate the relationship between the agents’ precision and the stochastic nature of disagreement, let us focus on the stochastic term in the dynamics of disagreement expressed in Equation (9). This term arises because $\gamma_A \neq \gamma_B$. As previously observed in Equation (5), different posterior variances are a result of different mean-reversion parameters. This generates stochastic shocks in disagreement. Although models of overconfidence (Scheinkman and Xiong, 2003; Dumas, Kurshev, and Uppal, 2009) generate a similar stochastic term, a key difference arises in our setup. To see this, suppose we shut down this stochastic term.

¹²The dynamics of disagreement in (9) comprise only \widehat{W}_B but not \widehat{W}_A . Without loss of generality, we choose to work under agent B ’s probability measure \mathbb{P}^B ; however, by using (8), we could easily switch to agent A ’s probability measure and all the results would still hold.

¹³Alternatively, if agent A believes the fundamental is persistent, then we can write the dynamics of disagreement under \mathbb{P}^A and the same intuition holds.

This can be done by properly adjusting the initial learning problem of the agents.¹⁴ Equation (9) shows that, even though the stochastic term disappears, disagreement will still be time-varying—and persistent—precisely due to the first term in its drift. In contrast, shutting down the stochastic term in models of overconfidence eliminates disagreement through prompt convergence toward its long-term mean, zero. This highlights the “structural” form of disagreement generated by different economic models.

2.2 Economic Outlook

Now, let us consider how model disagreement affects each agent’s relative economic *outlook*. Since each agent perceives the economy under a different probability measure, any random economic variable X , measurable and adapted to the observation filtration \mathcal{O} , now has two expectations: one under the probability measure \mathbb{P}^A , and the other under the probability measure \mathbb{P}^B . Naturally, they are related to each other by the formula

$$\mathbb{E}^A [X] = \mathbb{E}^B [\eta X],$$

where η measures the relative difference in outlook from one agent to the other.

Proposition 2. (*Economic Outlook*) *Under the probability measure \mathbb{P}^B , the relative difference in economic outlook satisfies*

$$\eta_t \equiv \left. \frac{d\mathbb{P}^A}{d\mathbb{P}^B} \right|_{\mathcal{O}_t} = e^{-\frac{1}{2} \int_0^t \left(\frac{1}{\sigma_\delta} \widehat{g}_s \right)^2 ds - \int_0^t \frac{1}{\sigma_\delta} \widehat{g}_s d\widehat{W}_{Bs}^\delta},$$

where \mathcal{O}_t is the observation filtration at time t and η obeys the dynamics

$$\frac{d\eta_t}{\eta_t} = -\frac{1}{\sigma_\delta} \widehat{g}_t d\widehat{W}_{Bt}^\delta. \tag{10}$$

Proof. See Girsanov’s Theorem. □

On the surface, the expression in (2) is simply the Radon-Nikodym derivative for the change of measure between the agents’ beliefs. But this has a natural economic interpretation here as the difference in economic outlook between the agents, since it captures the difference in expectations that each agent has for the future. This contrasts with previous papers that use η_t to express differences in the *sentiment* between agents (Dumas, Kurshev, and Uppal, 2009). In our setting, agents do not have behavioral biases like overconfidence or optimism.

¹⁴Precisely, we can consider that agents have different parameters σ_f chosen in such a way that $\gamma_A = \gamma_B$. This will shut down the stochastic term in Equation (9).

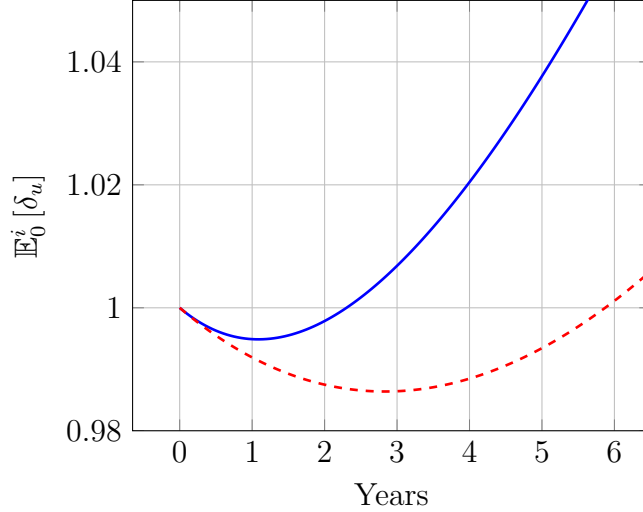


Figure 1: Different economic outlooks

Expectation of future aggregate consumption computed by agent A (solid blue line) and agent B (dashed red line). The state variables at time $t = 0$ are $\delta_0 = 1$, $\widehat{f}_{B0} = -1\%$, $\widehat{g}_0 = 0$, and $\eta_0 = 1$. Other parameters for this example are listed in Table 1.

Rather, because they adhere to different models of the world, they rationally have different economic outlooks, which are not a function of how they are feeling per se (i.e., sentiment).

One important implication of Proposition 2 is that disagreement over fundamentals and economic outlook are different entities. In fact, relative outlook is a function of disagreement, and there may be differences in the agents' outlook even though they agree today on the underlying fundamentals of the economy. This is because disagreement in our setup expresses the difference in beliefs about the expected growth rate *today*, while outlook enters into the expectations of future economic variables and thus captures the way in which agents' beliefs will differ into the *future*.

This is best appreciated by observing that the relative difference in outlook in (2) is a function of the integral of disagreement that is realized over a particular horizon, not just the disagreement that takes place at one particular instant. This implies that two agents may have very different outlooks, even though they currently agree on the fundamentals in the market. That is, even though their models currently yield the same fundamentals, because they use different models, they will have different outlooks for the future. Trading volume may therefore be substantial even when there is currently no disagreement about fundamentals: trade will still take place because the agents take into account that they will disagree in the future (i.e., they have different economic outlooks).

To see this more clearly, consider the following example. Suppose that, at $t=0$, $\widehat{f}_{A0} = \widehat{f}_{B0} = -1\%$. Because $\widehat{g}_0 = 0$, both agents agree that the economy is going through a recession. Furthermore, assume that agent B believes the economic cycles are longer than

agent A , that is, $\lambda_B = 0.1$ whereas $\lambda_A = 0.3$. Figure 1 shows the different economic outlooks that agents hold, *even though they are in agreement today*. It calculates the expectation of future dividends, $\mathbb{E}_0^i[\delta_u]$. Agent A (solid blue line) believes that the economy will recover quickly, in about two years, whereas agent B (dashed red line) believes that it will take six years for the economy to get back to its initial level of aggregate consumption.

It is also instructive to observe in (10) that disagreement affects the evolution of relative outlook. It is the primary driver of fluctuations in η_t . When \hat{g}_t is large, η_t will also have large fluctuations. Note, however, that even though $d\eta_t$ is zero when $\hat{g}_t = 0$, η_t itself can take any positive value and thus it still bears implications for the pricing of assets in the economy.

2.3 Equilibrium Pricing

To compute the equilibrium, we first write the optimization problem of each agent under agent B 's probability measure \mathbb{P}^B . Since we have decided to work (without loss of generality) under \mathbb{P}^B , let us write from now on and for notational ease the following conditional expectations operator

$$\mathbb{E}_t[\cdot] \equiv \mathbb{E}^B[\cdot | \mathcal{O}_t].$$

The market is complete in equilibrium since under the observation filtration of both agents there is a single source of risk. Consequently, we can solve the problem using the martingale approach of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989).¹⁵

Proposition 3. (*Equilibrium*) Assume that the coefficient of relative risk aversion α is an integer.¹⁶ The equilibrium price of the risky asset at time t is

$$S_t = \int_t^\infty S_t^u du,$$

where S_t^u is

$$S_t^u = \mathbb{E}_t \left[\frac{\xi_t^B}{\xi_u^B} \delta_u \right] = e^{-\rho(u-t)} \delta_t^\alpha \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega(\eta_t)^j [1 - \omega(\eta_t)]^{\alpha-j} \mathbb{E}_t \left[\left(\frac{\eta_u}{\eta_t} \right)^{\frac{j}{\alpha}} \delta_u^{1-\alpha} \right], \quad (11)$$

¹⁵The martingale approach transforms the dynamic consumption and portfolio choice problem into a consumption choice problem subject to a static, lifetime budget constraint.

¹⁶This assumption greatly simplifies the calculus. To the best of our knowledge, it has been first pointed out in Yan (2008) and Dumas, Kurshev, and Uppal (2009). If the coefficient of relative risk aversion is real, the computations can still be performed using Newton's generalized binomial theorem.

where ξ^B denotes the state-price density perceived by agent B

$$\xi_t^B = e^{-\rho t} \delta_t^{-\alpha} \left[\left(\frac{\eta_t}{\kappa_A} \right)^{1/\alpha} + \left(\frac{1}{\kappa_B} \right)^{1/\alpha} \right]^\alpha, \quad (12)$$

and $\omega(\eta)$ denotes agent A 's share of consumption

$$\omega(\eta_t) = \frac{\left(\frac{\eta_t}{\kappa_A} \right)^{1/\alpha}}{\left(\frac{\eta_t}{\kappa_A} \right)^{1/\alpha} + \left(\frac{1}{\kappa_B} \right)^{1/\alpha}}. \quad (13)$$

The risk free rate r and the market price of risk θ are

$$r_t = \rho + \alpha \widehat{f}_{Bt} - \alpha \omega(\eta_t) \widehat{g}_t + \frac{1}{2} \left[\frac{\alpha - 1}{\alpha \sigma_\delta^2} \omega(\eta_t) (1 - \omega(\eta_t)) \widehat{g}_t^2 - \alpha(\alpha + 1) \sigma_\delta^2 \right]$$

$$\theta_t = \alpha \sigma_\delta + \omega(\eta_t) \frac{\widehat{g}_t}{\sigma_\delta}.$$

Proof. The proof follows Dumas, Kurshev, and Uppal (2009) and is provided in Appendix A.4. The moment-generating function in Equation (11) is solved in Appendix A.5. \square

Equation (12) shows how the state-price density ξ^B depends on the outlook variable η . Since disagreement \widehat{g} directly drives the volatility of the state-price density (as shown in Equation 10), it follows from (12) that persistence in disagreement generates persistence in the volatility of the state-price density. Therefore, even though in our model agents disagree about a drift component, it directly impacts the diffusion of the state price density and consequently all the equilibrium quantities.

The optimal share of consumption, stated in Equation (13), is exclusively driven by the outlook variable η . If η tends to infinity, which means that agent A 's perception of the economy is more likely than agent B 's perception¹⁷, then agent A 's share of consumption tends to one. Conversely, if η tends to zero, then $\omega(\eta)$ converges to zero. Unsurprisingly, agent A 's consumption share increases with the likelihood of agent A 's probability measure being true.

The single-dividend paying stock, expressed in Equation (11), consists in a weighted sum of expectations, with weights characterized by the consumption share $\omega(\cdot)$, which itself is driven by the economic outlook η . It is instructive to study first the case $\alpha = 1$ (log-utility

¹⁷This can be seen from Equation (10): high η can arise either if (i) agent B is optimistic ($\widehat{g} > 0$) and \widehat{W}_B shocks are negative or if (ii) agent B is pessimistic ($\widehat{g} < 0$) and \widehat{W}_B shocks are positive.

case), when the price of the single-dividend paying stock becomes

$$S_t^u = \omega(\eta_t)S_{At}^u + [1 - \omega(\eta_t)]S_{Bt}^u,$$

where S_{it}^u is the price of the asset in a hypothetical economy populated by only group i agents. A similar aggregation result is provided by [Xiong and Yan \(2010\)](#). In contrast, when the coefficient of relative risk aversion is greater than one, the aggregation must be adapted to accommodate the additional intermediary terms (for $j = 1, \dots, \alpha - 1$) in the summation (11). In fact, the summation has now $\alpha + 1$ terms and the price of the single dividend paying stock becomes

$$S_t^u = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega(\eta_t)^j [1 - \omega(\eta_t)]^{\alpha-j} S_{jt}^u, \quad (14)$$

where S_{jt}^u is the price of the asset in a hypothetical economy populated by agents with relative economic outlook $\eta^{j/\alpha}$ ($j = 0$ corresponds to agent B and $j = \alpha$ corresponds to agent A). Since the binomial coefficients in (14) sum up to one, the price is therefore a weighted average of $\alpha + 1$ prices arising in representative agent economies populated by agents with relative economic outlook $\eta^{j/\alpha}$. Hence, the outlook variable η not only affects the price valuation through the expectations in (11), but also through the weights in the summation (14).

The weighted average form (14) highlights the origin of fluctuations in stock price volatility and the key role played by disagreement and the relative outlook η . The intuition is as follows. The relative outlook η fluctuates in the presence of disagreement and causes investors to speculate against each other. This speculative activity generates fluctuations in consumption shares: if the hypothetical investor j 's model is confirmed by the data, he or she will consume more and thus his or her weight in the pricing formula (14) increases. The price S_t^u will therefore approach S_{jt}^u not only through the expectation but also through changes in the relative weights. These fluctuations in relative weights further amplify the impact of disagreement on the stock price and thus generates excess volatility ([Dumas, Kurshev, and Uppal, 2009](#)).

3 Disagreement and Volatility

We turn now to the implications of model disagreement and different economic outlooks for the level, fluctuations, and persistence of volatility.

Proposition 4. (*Stock Return Volatility*) The time t stock return volatility satisfies

$$|\sigma_t| = \left| \frac{\sigma(X_t)^\top \frac{\partial S_t}{\partial X_t}}{S_t} \right| = \left| \frac{\sigma(X_t)^\top \int_t^\infty \frac{\partial S_t^u}{\partial X_t} du}{\int_t^\infty S_t^u du} \right|,$$

where $\sigma(x_t)$ denotes the diffusion of the state vector $x = (\zeta, \hat{f}_B, \hat{g}, \mu)$ and we define $\zeta \equiv \ln \delta$ and $\mu \equiv \ln \eta$. The stock return diffusion, σ_t , can be written

$$\sigma_t = \sigma_\delta + \underbrace{\frac{S_f \gamma_B}{S \sigma_\delta}}_{\equiv \sigma_{f,lr}} + \underbrace{\frac{S_g}{S} \left(\frac{\gamma_B - \gamma_A}{\sigma_\delta} \right)}_{\equiv \sigma_{g,lr}} + \underbrace{\frac{-S_\mu \hat{g}_t}{S \sigma_\delta}}_{\equiv \sigma_{g,i}}, \quad (15)$$

where S_f , S_g , and S_μ represent partial derivatives of stock price with respect to \hat{f}_B , \hat{g} .

Proof. The diffusion of the state vector $(\zeta, \hat{f}_B, \hat{g}, \mu)$ is obtained from Equations (6), (7), (9), and (10). Multiply these with $S_\zeta/S = 1$, S_f/S , S_g/S , and S_μ/S to obtain (15). \square

Equation (15) shows that the stock return diffusion σ consists in the standard Lucas (1978) volatility σ_δ and three terms representing the *long-run* impact of changes in the estimated fundamental \hat{f}_B (denoted by $\sigma_{f,lr}$), the *long-run* impact of changes in the disagreement \hat{g} (denoted by $\sigma_{g,lr}$), and the *instantaneous* impact of changes in the disagreement \hat{g} (denoted by $\sigma_{g,i}$). Since we assume the volatility of the dividend σ_δ to be constant, the volatility of the price-dividend ratio is exclusively driven by these last 3 terms. Therefore, all the following interpretations apply to both the stock return volatility and the volatility of the price-dividend ratio.

3.1 Dynamics of Volatility

The general consensus in the theoretical literature is that disagreement amplifies trading volume and produces excess volatility (see Harris and Raviv (1993), Banerjee and Kremer (2010), and Dumas, Kurshev, and Uppal (2009) among many others). Our model is no exception; in separate calculations, we show that the disagreement about the current growth rate and also the different economic outlooks that agents hold regarding the future both amplify volatility and trading volume. In this section, we turn our focus on more specific implications of our model regarding the *dynamics* of volatility.

We start by performing a decomposition of the volatility which helps us understand what drives its level and what drives its fluctuations. Then, we highlight two specific implications for which we find strong empirical support. First, we show both theoretically and empirically that the persistence of disagreement is indeed the main driver of the persistence of volatility.

Second, we show that volatility is strongly driven by disagreement especially in bad economic times (i.e., when the expected growth rate is low), whereas in good economic times the relation is weaker. This implication, too, finds support in the data.

The calibration that we use for our theoretical results is provided in Table 1. Parameters are adapted from Brennan and Xia (2001) and Dumas, Kurshev, and Uppal (2009), with a few differences. We choose lower values for the volatility of the fundamental and the dividend growth volatility. For the preference parameters, we choose a smaller coefficient of relative risk aversion and a positive subjective discount rate. The mean-reversion speed

Parameter	Symbol	Value
Relative Risk Aversion	α	3
Subjective Discount Rate	ρ	0.015
Agent A 's Initial Share of Consumption	ω_0	0.5
Consumption Growth Volatility	σ_δ	0.03
Mean-Reversion Speed of the Fundamental	λ_A	0.3
	λ_B	0.1
Long-Term Mean of the Fundamental	\bar{f}	0.025
Volatility of the Fundamental	σ_f	0.015

Table 1: Calibration

chosen by agent B is 0.1, corresponding to a business cycle half-life of approximately seven years. Agent B consequently believes in long-run risk. On the other hand, agent A , who chooses $\lambda_A = 0.3$, believes that the length of the business cycle is shorter with a perceived half-life of approximately two years. We assume that the true λ lies somewhere in between λ_B and λ_A , and thus neither agent has a superior learning model.

3.1.1 Level and variation of volatility

As can be seen from Proposition 4, a direct analysis of the stock diffusion formula (15) is obscured by the presence of the partial derivatives S_f , S_g , and S_μ . These derivatives depend on the state variables themselves and thus are time-varying. In order to gain more intuition and to understand which terms drive the level of volatility and which ones drive its fluctuations, we simulate the last three terms in Equation (15). Simulations are done at weekly frequency for 100 years. Figure 2 illustrates one simulated path of the stock return diffusion and its components. The significant driver of changes in stock market volatility is the fourth term in Equation (15), $\sigma_{g,i}$, whereas terms representing long-run changes in disagreement, $\sigma_{g,lr}$, and long-run changes in the estimated fundamental, $\sigma_{f,lr}$, are slightly time-varying but have less significant impact on the dynamics of volatility.

The fourth term in Equation (15) is therefore key to understanding the impact of disagreement on stock return volatility. This term consists in the partial derivative of the stock

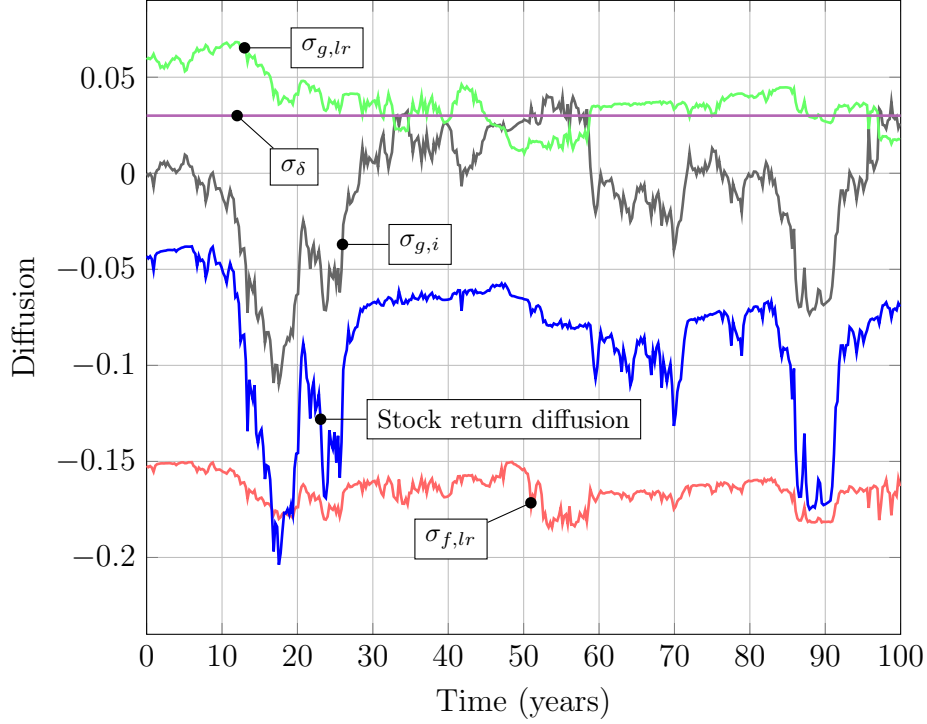


Figure 2: Stock return diffusion and its components

One simulated path (100 years) of the stock return diffusion and its components. Simulations are performed at weekly frequency, but lines are plotted at quarterly frequency to avoid graph cluttering. The diffusion components $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$ are defined in Equation (15). The calibration is provided in Table 1.

price with respect to the relative outlook variable η , multiplied by the volatility of η (which itself is directly driven by disagreement—according to 10). Both disagreement and economic outlooks therefore play a role in driving volatility, by the following mechanism. When agents are in disagreement, they hold different economic outlooks and thus the stock price fluctuates in order to accommodate speculative trading by both agents. Higher disagreement generates large fluctuations in economic outlook (according to 10) and thus large changes in the stock price.

To disentangle the role played by disagreement from the role played by the relative economic outlook, we plot in the left panel of Figure 3 the fourth diffusion component $\sigma_{g,i}$ and the disagreement \hat{g} . The correlation coefficient between the two lines in this particular example yields a value of 0.95. In the right panel of Figure 3 we plot the distribution of the correlation between the diffusion term and disagreement for 1,000 simulations and we find that the coefficient stays mainly between 0.8 and 1. It is therefore disagreement which drives the fluctuations in $\sigma_{g,i}$, whereas the relative economic outlook is the primary channel through which these fluctuations are transmitted to stock market volatility.

We examine whether the dynamics illustrated on Figure 2 are particular to one simulation.

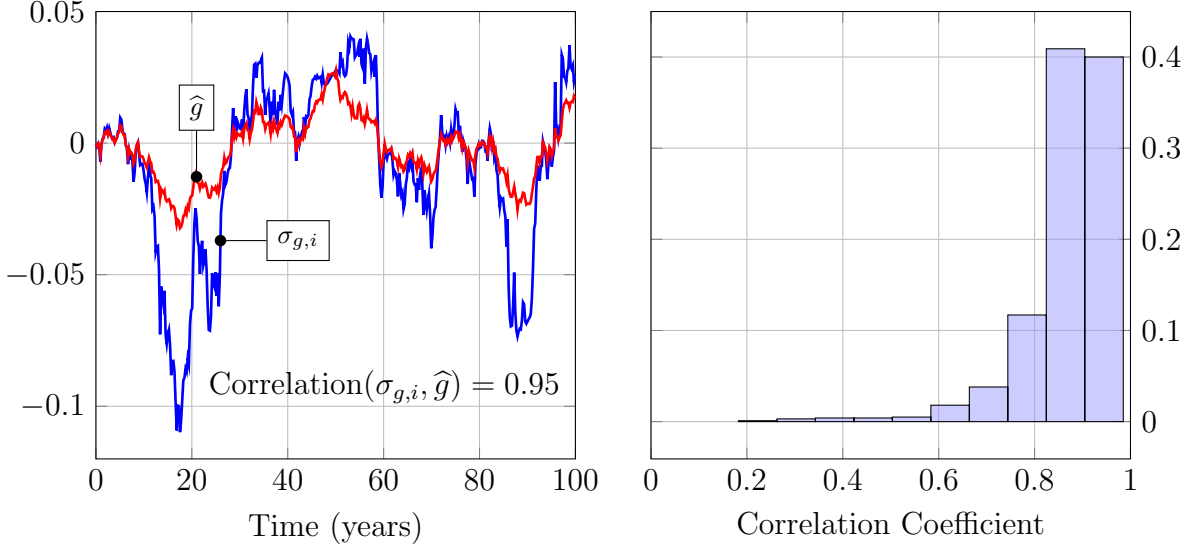


Figure 3: Volatility component $\sigma_{g,i}$ and disagreement \hat{g}

The left panel depicts one 100 years simulation of the volatility component $\sigma_{g,i}$ and the associated disagreement \hat{g} . Simulations are performed at weekly frequency, but lines are plotted at quarterly frequency to avoid graph cluttering. The volatility term $\sigma_{g,i}$ is defined in Equation (15). The calibration is provided in Table 1. The right panel shows the distribution of the correlation between $\sigma_{g,i}$ and \hat{g} . This correlation is computed over an horizon of 100 years (simulated at weekly frequency), for 1,000 simulations.

To this end, we plot in Figures 4 and 5 the distributions of the averages and variances of $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$. Averages and variances are computed over the length of each simulation which is chosen to be 100 years at weekly frequency.

Figure 4 shows that the diffusion components $\sigma_{g,lr}$ and $\sigma_{g,i}$ do not have a significant impact on the level of volatility. The level of volatility is primarily determined by the \hat{f}_B -term defined by $\sigma_{f,lr}$. It is worth mentioning that the \hat{f}_B -term is negative because in our model the precautionary savings effect dominates the substitution effect. Indeed, a positive shock in the fundamental increases future consumption. Because agents want to smooth consumption over time, they increase their current consumption and so reduce their current investment. This tendency to disinvest outweighs the substitution effect (according to which agents would invest more) and implies a drop in prices as long as agents are sufficiently risk averse ($\alpha > 1$). Hence the stock return diffusion component determined by changes in the fundamental, $\sigma_{f,lr}$, is negative. The smaller the mean-reversion speed λ_B , the more negative the $\sigma_{f,lr}$ component, and consequently the larger stock return volatility becomes: a small mean-reversion speed implies a significant amount of long-run risk and therefore the stock price is very sensitive to movements in the fundamental, as in Bansal and Yaron (2004).

We try now to understand which components drive the variability of stock return diffusion. This is shown in Figure 5, which depicts the variances of the diffusion components and

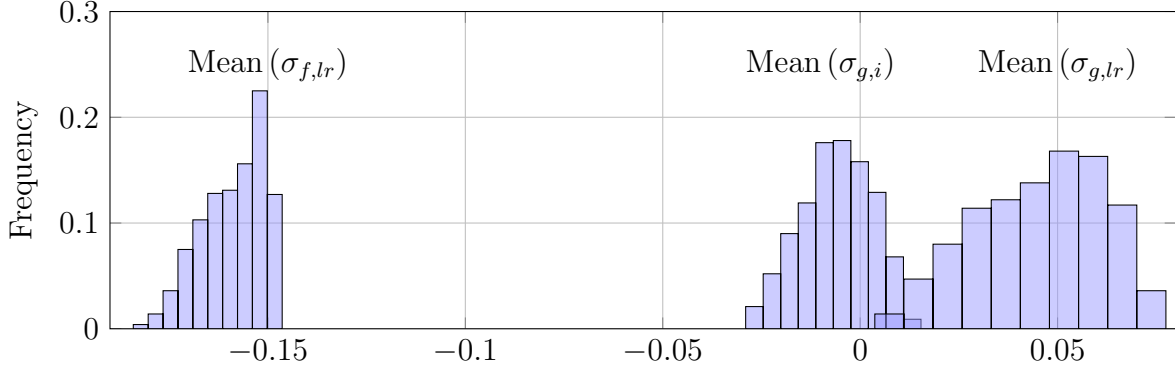


Figure 4: Distribution of the average of the diffusion components

The average over 1,000 simulations of each of the last three diffusion components in Equation (15) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.

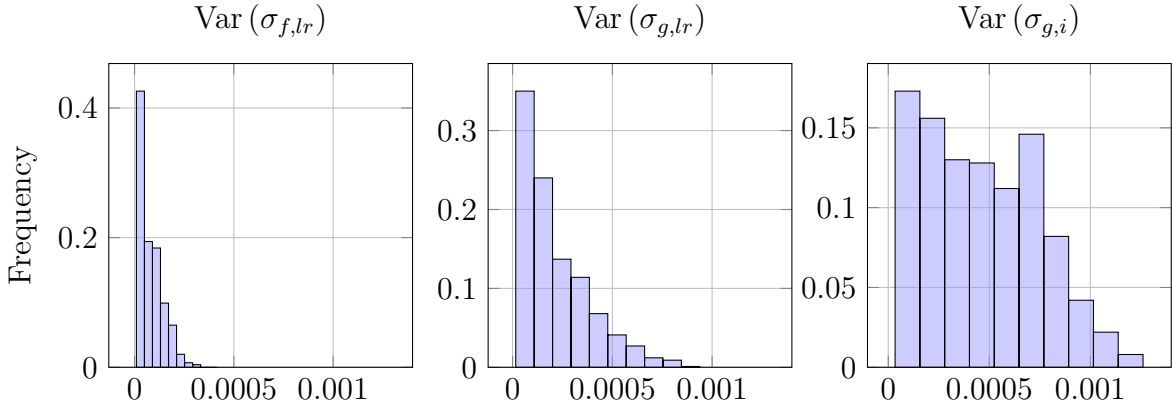


Figure 5: Distribution of the variance of the diffusion components

The variance over 1,000 simulations of each of the last three diffusion components in Equation (15) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.

confirms the conclusions drawn from the example depicted in Figure 2. Variations incurred by the stock return diffusion are almost exclusively generated by variations in the third and fourth diffusion terms, $\sigma_{g,lr}$ and $\sigma_{g,i}$, which are both driven by disagreement. Indeed, variations in $\sigma_{f,lr}$ are relatively small. We can therefore conclude that the level of the volatility is mainly driven by the persistence of the expected consumption growth, whereas fluctuations in volatility are driven by differences of beliefs regarding the persistence of the expected consumption growth.

3.1.2 Persistence of volatility

We turn now to the question whether the fluctuations in volatility generated by disagreement are persistent. We show that indeed, in the model stock return volatility clusters because

of the following mechanism. As shown in Proposition 1, disagreement \widehat{g} mean-reverts to a stochastic mean driven by \widehat{f}_B . Because one of the agents (in this case agent B) believes the fundamental is persistent, agent B 's estimation of the fundamental \widehat{f}_B is persistent and so becomes the disagreement. Given that the disagreement enters the diffusion of state-price density through the outlook variable η (see Proposition 2) and then enters volatility through the last component in Equation (15), stock return volatility clusters. This mechanism, new to our knowledge, shows how persistence in the fundamental (a component of the *drift*) can transmute into the *diffusion* of stock returns and generate volatility clustering.

To provide evidence that persistent disagreement indeed implies GARCH-type dynamics in our theoretical model, we simulate 1,000 paths of stock returns over a 100 years horizon at weekly frequency. For each simulated path we compute the demeaned returns, ϵ , by extracting the residuals of the $AR(1)$ regression

$$r_{t,t+1} = \alpha_0 + \alpha_1 r_{t-1,t} + \epsilon_{t+1},$$

where $r_{t,t+1}$ stands for the stock return between time t and $t + 1$. The demeaned returns ϵ is then fitted to a GARCH(1,1) process defined by

$$\begin{aligned} \epsilon_t &= \sigma_t z_t, \text{ where } z_t \sim N(0, 1) \\ \sigma_{t+1}^2 &= \beta_0 + \beta_1 \epsilon_t^2 + \beta_2 \sigma_t^2. \end{aligned}$$

Figure 6 illustrates the distribution of the ARCH parameter β_1 and the GARCH parameter β_2 . Their associated t -statistics range between 6 and 11 for the ARCH parameter and between 150 and 350 for the GARCH parameter. The values of β_1 , β_2 , and in particular their sum, show therefore that stock return volatility clusters and is close to be integrated. That is, the model implied volatility clusters because its main driver—the disagreement among agents—is persistent.

The prediction that the persistence in disagreement drives the persistence in volatility can be tested empirically. For this purpose, we use the dispersion of analyst forecasts of the 1-quarter-ahead real GDP growth rate and the annualized daily volatility of S&P 500 returns over each quarter, between Q4:1968 and Q2:2014, as proxies for absolute disagreement and volatility. Then, we perform 5-year rolling-window regressions of disagreement and volatility on their respective lagged values and we use the associated autocorrelation coefficients to measure the persistence in both series. Figure 7 plots these autocorrelation coefficients both in a scatter plot (left panel) and in time-series (right panel). The overall message of both panels is that the persistence in volatility and the persistence in dispersion feature similar levels and dynamics. Moreover, the two time-series display an evident positive correlation.

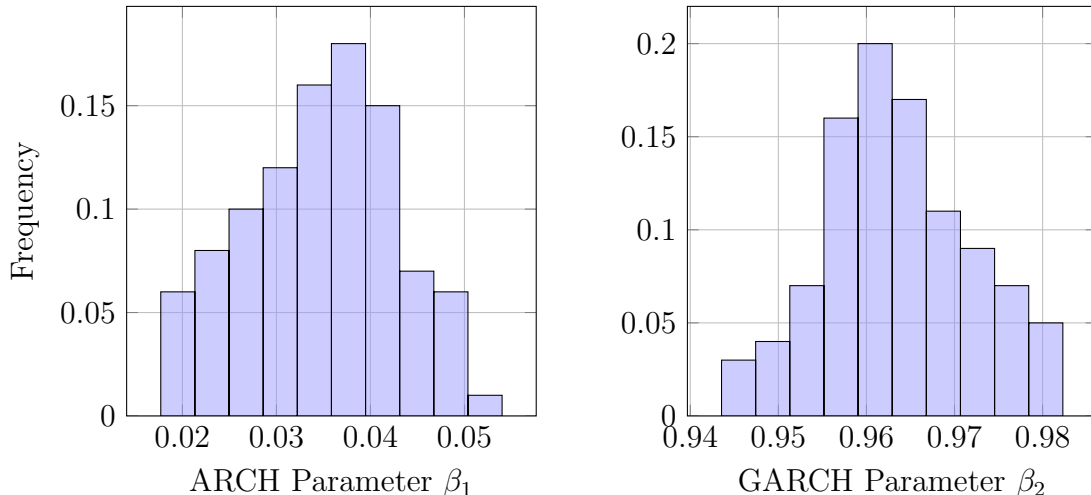


Figure 6: Model implied ARCH and GARCH parameters

Volatility the Distribution of the ARCH and GARCH parameters, resulted from 1,000 simulations over 100 years, at weekly frequency. The calibration is provided in Table 1.

Regression results in Table 2 confirm that the persistence in dispersion is significantly and positively correlated with the persistence in volatility. Column 2 shows the fitted regression line depicted in the left panel of Figure 7, that is, the regression of the persistence of volatility on a constant and the persistence of disagreement. In columns 3 and 4 we perform the same regression, but instead of using the autocorrelation coefficient as measure of persistence, we use the AR(1) coefficients of the rolling-window regressions (column 3) or their t -statistics (column 4). Overall, the results are in favor of our theoretical prediction, namely that the persistence in disagreement drives the persistence in volatility.¹⁸

3.1.3 The asymmetric relationship between disagreement and volatility

In our model, the relation between disagreement and volatility depends on other state variables. Precisely, depending on whether agents believe the economy is going through good or bad times, disagreement has a different impact on volatility. To see this, we plot in Figure 8 the stock return volatility as a function of the disagreement for different values of the fundamental (as estimated by agent B , i.e., \hat{f}_B). Because agent B believes the fundamental reverts slower to its mean than agent A ($\lambda_B < \lambda_A$), agent B tends to be optimistic when the fundamental is large and pessimistic when the fundamental is low. Therefore, the disagreement tends to be positive when the fundamental is large (in good economic times) and

¹⁸To check the robustness of these results, we have considered different lengths of the rolling window (from 1 to 20 years). We have also used the three available measure of cross-sectional dispersion available (i.e., dispersion for levels (D1), dispersion for Q/Q growth (D2), or dispersion for log difference of levels (D3)), with similar results. We also obtain similar results if we consider analyst forecasts several quarters ahead.

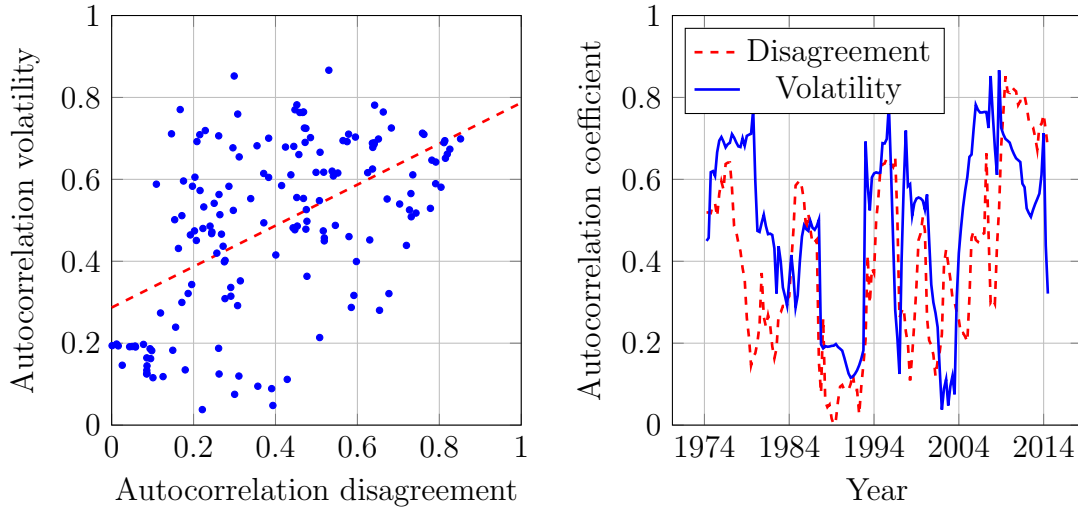


Figure 7: Persistence of disagreement vs. Persistence of volatility

We use the dispersion of analyst forecasts of the 1-quarter ahead GDP growth rate as a proxy for disagreement and the annualized daily volatility over each quarter as a proxy for volatility. Persistence is measured by means of the autocorrelation coefficient obtained by performing 5-year rolling-window regressions of the variable of interest (disagreement or volatility) on its lagged value. The left panel depicts the scatter plot of the autocorrelation coefficients and the fitted linear relationship (see Table 2 for the regression coefficients). The right panel depicts the time-series of the autocorrelation coefficients. Data points are at quarterly frequency from Q2:1974 to Q2:2014. We de-trend the analysts’ dispersion time-series because it features a strong decreasing linear trend.

	Autocorrelation coefficients	AR(1) coefficients	<i>t</i>-stats of AR(1) coeffs.
Intercept	0.287*** (4.527)	0.307*** (6.096)	1.837*** (5.073)
Persist. Disagr _{<i>t</i>}	0.500*** (4.510)	0.478*** (5.241)	0.415*** (4.047)
<i>R</i> ²	0.287	0.256	0.217
Obs	162	162	162

Table 2: Persistence of disagreement vs. Persistence of volatility

Regressions of the persistence of S&P 500 volatility on the persistence of disagreement. We measure persistence in three ways: (i) autocorrelation coefficients as in Figure 7 (column 2), (ii) AR(1) coefficients of the regression of each variable on its lagged values (column 3), and (iii) *t*-stats of the AR(1) coefficients (column 4). Data points are at quarterly frequency from Q4:1968 to Q2:2014. We de-trend the analysts’ dispersion time-series because it features a strong decreasing linear trend. *t*-statistics are reported in brackets and statistical significance at the 10%, 5%, and 1% levels is labeled *, **, and ***, respectively. Because residuals are significantly autocorrelated, *t*-stats are adjusted using the Newey and West (1987) procedure.

negative when the fundamental is low (in bad economic times).¹⁹ In Figure 8, the solid blue

¹⁹Equations (7) and (9) show that, indeed, the correlation between the disagreement and the fundamental is equal to 1. Therefore, disagreement is most likely positive in good times (when the fundamental is large) and negative in bad times (when the fundamental is small).

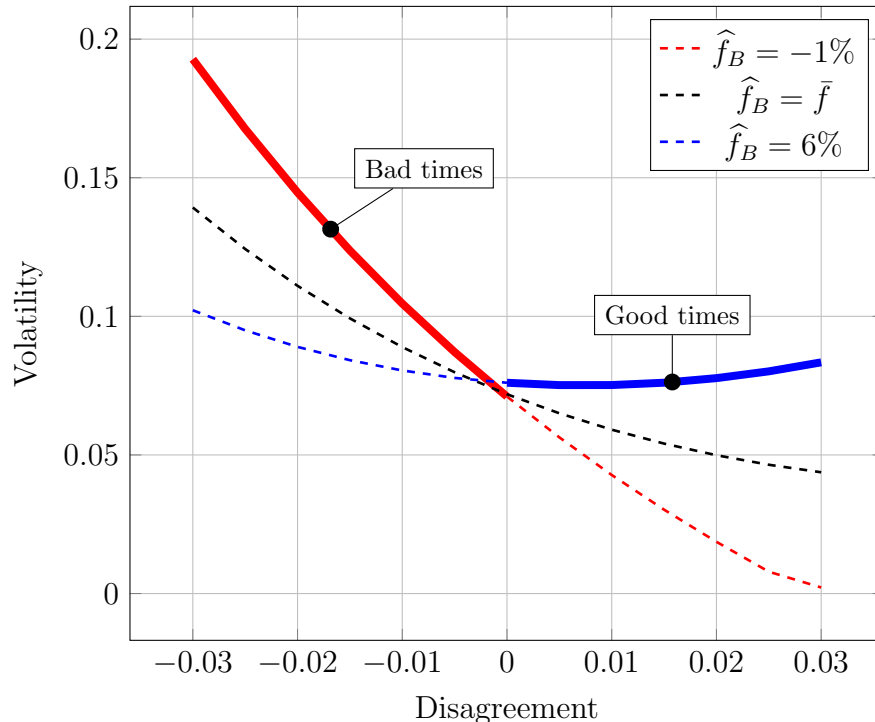


Figure 8: Stock return volatility vs. disagreement

The red, black, and blue curves depict the relation between volatility and disagreement when the fundamental is $\hat{f}_B = -1\%$, $\hat{f}_B = \bar{f}$, and $\hat{f}_B = 6\%$, respectively. The solid red and blue curves illustrate the most likely relation applying in bad times and good times, respectively. The calibration is provided in Table 1.

curve depicts the most likely relationship between volatility and disagreement in good times, whereas the solid red curve depicts the relationship in bad times. We observe a “hockey-stick pattern,” i.e., a strong positive relation between *absolute disagreement* and volatility in bad times and a considerably weaker relation in good times.²⁰

The prediction that absolute disagreement and return volatility are strongly and positively related in bad times, but not significantly related in good times, is empirically testable. To implement the test, we use the dispersion of analyst forecasts of the 1-quarter-ahead real GDP growth rate and the annualized daily volatility of S&P 500 returns over each quarter between Q4:1968 and Q2:2014 as proxies for absolute disagreement and volatility. Then, we perform a predictive regression of 1-quarter-ahead volatility on the current disagreement, in which the slope coefficients depend on the state of the economy. The expansion dummy, denoted I_t , is 1 if the U.S. economy is in expansion in the same quarter. We consider two

²⁰This can also be seen by considering the quadratic relation between stock return variance and disagreement. From Equation (15), the stock return variance can be written: $\sigma_t^2 = A_{1t}\hat{g}_t^2 + A_{2t}\hat{g}_t + A_{3t}$, where the loading $A_{1t} = (S_\mu/S\sigma_\delta)^2 > 0$ is a decreasing function of the fundamental (a pattern that we have identified numerically). Consequently, when the fundamental is low, the loading is high and thus a “hockey-stick” pattern emerges (we have also run simulations and found the same pattern: the relation between negative disagreement and volatility is stronger than the relation between positive disagreement and volatility).

definitions of good economic times: (i) we define good times as NBER expansions, and (ii) we define good times as quarters where the mean forecast (across analysts) is larger than its average (over the time-series). The results thus correspond to the following regression:

$$\text{Volatility}_{t+1} = a + (b + b^{\text{exp}} \times I_t)\text{Disagreement}_t + \epsilon_{t+1}. \quad (16)$$

The coefficient b in (16) is interpreted as the sensitivity of volatility to disagreement during bad times, whereas the sum of coefficients $b + b^{\text{exp}}$ is interpreted as the sensitivity of volatility to disagreement during good times.

Table 3 reports the regression results. The second column reports results obtained when excluding the expansion dummies. The third and fourth column report the full regression specification (16), using our two definitions of good economic times. Consistent with the prediction of our model and irrespective of the definition of good times, Table 3 shows that the relation between volatility and dispersion is stronger in bad times than in good times. That is, the loadings that apply in bad times (coefficients b in columns 3 and 4) are positive, large, and significant, whereas in good times the loadings (sum of coefficients $b + b^{\text{exp}}$ in columns 3 and 4) are positive, but not statistically significant: their t -statistics are 0.449 and 0.776 respectively. Thus, although unconditionally volatility increases with disagreement (column 2 shows a weakly significant coefficient), there is a strong asymmetric relation between volatility and dispersion over the business cycle. As in our theoretical model, this relation features a “hockey-stick” pattern.

3.1.4 Volatility clustering in alternative theoretical models

We conclude this section with two questions. First, can a single agent framework generate volatility clustering? Second, if agents’s difference of beliefs is generated by overconfidence instead of model disagreement (Scheinkman and Xiong, 2003; Dumas, Kurshev, and Uppal, 2009), can we also observe volatility clustering?

To address the first question, we observe that in a single agent model the last two terms in Equation (15) disappear and volatility depends only on σ_δ and $\sigma_{f,lr}$. The analysis above indicates that this second term does not move significantly. Therefore, without disagreement there are no significant fluctuations in volatility and thus single agent models have difficulties in generating volatility clustering.

Turning now to the second question, Proposition 1 shows that disagreement mean-reverts around a persistent \hat{f}_B and thus itself becomes persistent. In contrast, disagreement generated by overconfidence, as in Dumas, Kurshev, and Uppal (2009), is not easily persistent, even though both agents would be long-term believers. The reason is that disagreement mean-reverts around zero with a parameter equal to $\lambda + \gamma/\sigma_\delta^2$ (see Lemma 2 in Dumas,

		$I_t = 1$ iff $t \in \text{NBER Expansion}$	$I_t = 1$ iff $\tilde{f}_t > \text{Average}(\tilde{f})$
a	0.150*** (25.788)	0.147*** (24.619)	0.149*** (24.890)
b	1.673* (1.803)	5.603** (2.528)	2.514** (2.514)
b^{exp}		-4.808* (-1.950)	-1.882 (-0.982)
R^2	0.018	0.038	0.023
Obs	183	183	183

Table 3: Relation between S&P 500 volatility and dispersion among forecasters

Predictive regressions of future volatility on lagged disagreement. Volatility is measured as the annualized daily volatility of S&P 500 over each quarter. Disagreement is measured as the analysts' forecast dispersion about the one-quarter-ahead US GDP growth. \tilde{f}_t stands for the mean (across analysts) analyst forecast on the GDP growth rate at time t , $\text{Average}(\tilde{f})$ for the average (over the entire time-series) of the mean analyst forecast, Disagr_t for the dispersion among analysts at time t , and N for the number of observations. The indicator I_t equals 1 in good times and 0 in bad times. Data are at quarterly frequency from Q4:1968 to Q2:2014. We de-trend the analysts' dispersion time-series because it features a strong decreasing linear trend. t -statistics are reported in brackets, and statistical significance at the 10%, 5%, and 1% levels is labeled *, **, and ***, respectively.

Kurshev, and Uppal 2009). Because of the second term, this parameter is large under usual calibrations, which is not enough to generate persistent dynamics.

On a final note, in Appendix A.6, Table 7, we perform a robustness analysis which further confirms the role played by disagreement for generating persistence and fluctuations in volatility. The analysis consists in comparing the properties of the model implied volatility for different calibrations. Consistent with our theoretical results, we find that strong long-run risk increases the average level of volatility, while severe disagreement increases both the variation and the persistence of volatility.

4 Survival

In our model we make the assumption that the fundamental is unobservable. It is consequently reasonable to assume that both investors have different beliefs regarding the dynamics of an unobservable process. Furthermore, it would be arbitrary and non-realistic to assume that one of the two agents has the correct beliefs i.e. the right model in mind. This raises two questions: what is the true data-generating process and how long do agents survive given this true data-generating process? This section is devoted to a discussion of these two questions.

In order to investigate how long each agent survives, we have to assume a realistic data-

generating process in the sense that it has to be consistent with agents beliefs. Indeed, although both agents might realistically have wrong beliefs, they agents are not too far from the truth. Therefore, we assume that the true data-generating process is

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma dW_t^\delta \quad (17)$$

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f, \quad (18)$$

where W^δ and W^f are two independent Brownian motions under the true probability measure \mathbb{P} . The true mean-reversion speed λ is assumed to be the average of agents A and B estimated mean-reversion speeds. Table 4 provides the values of the true and perceived mean-reversion speeds as well as their corresponding half-lives. Both agents misperceive the true length of the business cycle, but one overestimates it whereas the other underestimates it.

Belief	λ	Half-Life
Agent A	0.3	2.31
True value	0.2	3.46
Agent B	0.1	6.93

Table 4: Mean-reversion speed and length of the business cycle

To investigate the speed at which agent A or agent B disappears from the economy, we follow Yan (2008) and compute the \mathbb{P} -expectation of the consumption share of agent A . To this end we first simulate the dividend process using the true data-generating process provided in (17) and (18). Then, we perform each agent’s learning exercise and we compute the expectation of the consumption share of agent A , $\omega(\eta_T)$, for T ranging from 0 to 1,000 years.

Figure 9 depicts the \mathbb{P} -expectation (i.e., under the true probability measure) of the consumption share of agent A over 1,000 years. This expected consumption share slightly decreases on average. Investors who believe in long-run risk are expected to save more and thus have a lower survival index (Yan, 2008). Nevertheless, Figure 9 shows that both shares of consumption remain very close to each other and that both agents survive for more than 1,000 years. This is also consistent with Yan (2008). We conclude therefore that the type of disagreement considered here is economically important over long horizons.

5 Conclusion

We consider a theoretical framework in which two agents interpret information using different economic models of the economy. Specifically, in our setup agents disagree on the length of the business cycle. We analyze the asset pricing implications of such disagreement.

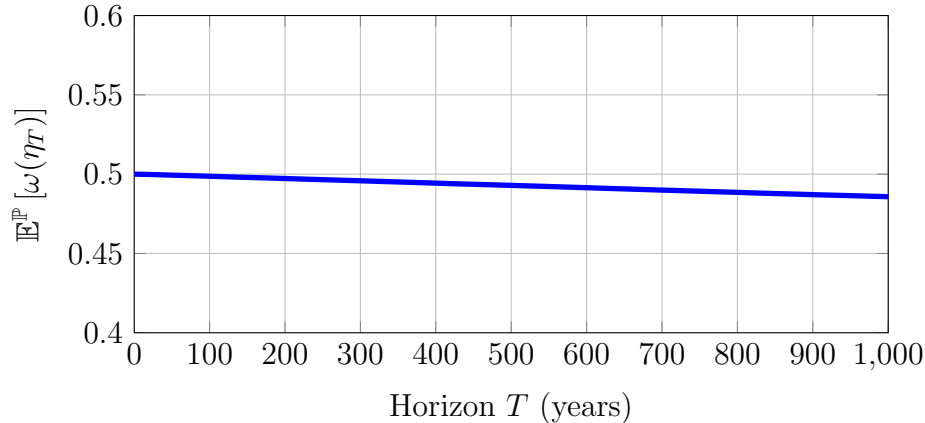


Figure 9: Expected share of consumption of agent A

Expectation under the true probability measure \mathbb{P} of the consumption share of agent A, $\omega(\eta_T)$. 10,000 simulations over 1,000 years are performed. The calibration is provided in Table 1.

We first show that the disagreement is strongly persistent, affects the volatility of the stochastic discount factor and consequently impacts the stock return volatility. Disagreement generates different expectations of future economic variables, or different economic outlooks, which further amplify the return volatility and the trading volume. We decompose the dynamics of volatility and show that disagreement is the main driver of volatility fluctuations, while the absolute level of volatility is driven primarily by long-run risk. We thus provide a theoretical foundation of the GARCH-like behavior of stock returns. We further document empirically a strong link between the persistence of disagreement and the persistence of volatility and also a strong relationship between disagreement and volatility in bad economic times, consistent with the main predictions of our model.

Several questions are the subject of our ongoing research. First, we assume that investors do not change their economic models. It is important to understand how our results would change if agents were to perform the full learning exercise. Our expectation is that investors' estimates should end up close to the true model only after a very long time. Indeed, an accurate estimator of the mean-reversion speed of a relatively persistent process requires a large sample of data (Hansen, Heaton, and Li, 2008). In addition, the large set of plausible models governing the real economy makes it virtually impossible for the agents to end up in agreement.

The learning uncertainty is by construction constant in our setup (see David (2008) for a model in which uncertainty is fluctuating). This is because we do not consider here any additional news (newspapers, quarterly reports, economic data, and so on). In a setting with additional news and in which investors' attention to news is fluctuating, uncertainty will fluctuate. Our conjecture is that spikes in attention will exacerbate the disagreement

among agents, further amplifying the effects on volatility described in this paper. It is therefore important to study the synergistic relationships between attention, uncertainty, and disagreement and their impact on asset prices.

Finally, our model generates a term structure of disagreement whose shape is governed by the difference between the mean-reversion parameters. Empirically observed term structures of disagreement (as in [Patton and Timmermann \(2010\)](#) or [Andrade, Crump, Eusepi, and Moench \(2014\)](#)) can therefore help estimating the magnitude of the difference between these parameters. The term structure of disagreement should also have implications on the pricing of firms with different characteristics and probably explain well-known anomalies such as the value premium.

A Appendix

A.1 Kalman/Maximum-Likelihood vs. Particle Filtering

Let us assume that the true data-generating process satisfies

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dW_t^\delta \quad (19)$$

$$df_t = \lambda(\bar{f} - f_t)dt + \sigma_f dW_t^f, \quad (20)$$

where W^δ and W^f are independent Brownian motions. The true parameters defining the dynamics of the dividend δ and the fundamental f are provided in Table 5.

Parameter	Symbol	Value
Dividend Growth Volatility	σ_δ	0.03
Mean-Reversion Speed of the Fundamental	λ	0.2
Long-Term Mean of the Fundamental	\bar{f}	0.025
Volatility of the Fundamental	σ_f	0.015

Table 5: True parameters

We simulate dividend data at quarterly frequency over a 50-year horizon²¹ using the true data-generating process defined in Equations (19) and (20) and the parameters provided in Table 5. Each agent uses the quarterly dividend data to estimate the following discrete-time model

$$\log\left(\frac{\delta_{t+\Delta}}{\delta_t}\right) = \left(f_t - \frac{1}{2}\sigma_\delta^2\right)\Delta + \sigma_\delta\sqrt{\Delta}\epsilon_{1,t+\Delta}$$

$$f_{t+\Delta} = A_f f_t + B_f + C_f \epsilon_{2,t+\Delta},$$

where $A_f = e^{-\lambda\Delta}$, $B_f = \bar{f}(1 - e^{-\lambda\Delta})$, $C_f = \frac{\sigma_f}{\sqrt{2\lambda}}\sqrt{1 - e^{-2\lambda\Delta}}$, and ϵ_1, ϵ_2 are independent Gaussian random variables with mean 0 and variance 1.

Although agents have the same information at hand, we assume that they use different econometrics techniques to perform their estimation exercise. Agent *A* estimates the unobservable fundamental and the parameters by applying the Kalman filter together with Maximum-Likelihood (Hamilton, 1994), while agent *B* applies the particle filtering algorithm presented in Liu and West (2001).²²

Table 6 shows that agent *A* and *B* obtain parameter estimates of the dividend volatility σ_δ , the long-term mean of the fundamental \bar{f} , and the volatility of the fundamental σ_f that are relatively close to each other. Their estimation of the mean-reversion speed λ , however, differs significantly from one another. Indeed, the absolute difference between the mean-reversion speed estimated by agent *A* and that estimated by agent *B* is worth 0.1742. Relative to the true value of the parameter, the difference in the estimated mean-reversion speeds is about 87%, whereas it is less than 25% for all other parameters. Therefore, this calibration exercise motivates, first, our assumption to consider heterogeneity in mean-reversion speeds only and, second, our choice to consider mean-reversion speeds such that $|\lambda_A - \lambda_B| = 0.2$.

²¹The frequency and horizon considered match those of the Real GDP growth time-series available on the Federal Reserve Bank of Philadelphia's website.

²²We would like to thank Arthur Korteweg and Michael Rockinger for providing us with various particle filtering codes. The particle filtering algorithm of Liu and West (2001) estimates, at each point in time, the unobservable fundamental and the parameters of the model.

Definition	Symbol	Value
Absolute Difference in Dividend Growth Volatility	$ \sigma_{A\delta} - \sigma_{B\delta} $	0.0022
Absolute Difference in Mean-Reversion Speed of the Fundamental	$ \lambda_A - \lambda_B $	0.1742
Absolute Difference in Long-Term Mean of the Fundamental	$ \bar{f}_A - \bar{f}_B $	0.0056
Absolute Difference in Volatility of the Fundamental	$ \sigma_{Af} - \sigma_{Bf} $	0.0032
Relative Difference in Dividend Growth Volatility	$\frac{ \sigma_{A\delta} - \sigma_{B\delta} }{\sigma_\delta}$	0.0749
Relative Difference in Mean-Reversion Speed of the Fundamental	$\frac{ \lambda_A - \lambda_B }{\lambda}$	0.8683
Relative Difference in Long-Term Mean of the Fundamental	$\frac{ \bar{f}_A - \bar{f}_B }{f}$	0.2254
Relative Difference in Volatility of the Fundamental	$\frac{ \sigma_{Af} - \sigma_{Bf} }{\sigma_f}$	0.2105

Table 6: Estimated parameters: maximum-likelihood vs. particle filter

Agent A applies the Kalman-filter together with Maximum-Likelihood, while agent B applies the particle filter algorithm of Liu and West (2001). The parameter values $\sigma_{B\delta}$, λ_B , \bar{f}_B , and σ_{Bf} are those obtained at the terminal time. Numbers reported above are medians computed over 1,000 simulations.

A.2 Filtering Problem

Agent A 's learning problem

Following the notations of Liptser and Shiryaev (2001), the observable process is

$$\begin{aligned} \frac{d\delta_t}{\delta_t} &= (A_0 + A_1 f_{At})dt + B_1 dW_{At}^f + B_2 dW_{At}^\delta \\ &= (0 + 1 \cdot f_{At})dt + 0 \cdot dW_{At}^f + \sigma_\delta dW_{At}^\delta. \end{aligned}$$

The unobservable process f_A satisfies

$$\begin{aligned} df_{At} &= (a_0 + a_1 f_{At})dt + b_1 dW_{At}^f + b_2 dW_{At}^\delta \\ &= (\lambda_A \bar{f} + (-\lambda_A) f_{At})dt + \sigma_f dW_{At}^f + 0 \cdot dW_{At}^\delta. \end{aligned}$$

Thus,

$$\begin{aligned} bob &= b_1 b'_1 + b_2 b'_2 = \sigma_f^2 \\ BoB &= B_1 B'_1 + B_2 B'_2 = \sigma_\delta^2 \\ boB &= b_1 B'_1 + b_2 B'_2 = 0. \end{aligned}$$

The estimated process defined by $\widehat{f}_{At} = \mathbb{E}^{\mathbb{P}^A}(f_{At}|\mathcal{O}_t)$ has dynamics

$$d\widehat{f}_{At} = (a_0 + a_1\widehat{f}_{At})dt + (boB + \gamma_{At}A'_1)(BoB)^{-1}\left(\frac{d\delta_t}{\delta_t} - (A_0 + A_1\widehat{f}_{At})dt\right),$$

where the posterior variance γ_{At} solves the ODE

$$\dot{\gamma}_{At} = a_1\gamma_{At} + \gamma_{At}a'_1 + bob - (boB + \gamma_{At}A'_1)(BoB)^{-1}(boB + \gamma_{At}A'_1)'$$

Assuming that we are at the steady-state yields

$$a_1\gamma_{At} + \gamma_{At}a'_1 + bob - (boB + \gamma_{At}A'_1)(BoB)^{-1}(boB + \gamma_{At}A'_1)' = 0.$$

Consequently,

$$d\widehat{f}_{At} = \lambda_A(\bar{f} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}_{At}^\delta$$

where

$$\begin{aligned}\gamma_A &= \sqrt{\sigma_\delta^2(\sigma_\delta^2\lambda_A^2 + \sigma_f^2)} - \lambda_A\sigma_\delta^2 \\ d\widehat{W}_{At}^\delta &= \frac{1}{\sigma_\delta} \left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{At}dt \right).\end{aligned}$$

Agent B 's learning problem

The estimated process is defined by $\widehat{f}_{Bt} = \mathbb{E}^B(f_{Bt}|\mathcal{O}_t)$. Doing the same computations as before yields

$$d\widehat{f}_{Bt} = \lambda_B(\bar{f} - \widehat{f}_{Bt})dt + \frac{\gamma_B}{\sigma_\delta}d\widehat{W}_{Bt}^\delta,$$

where

$$\begin{aligned}\gamma_B &= \sqrt{\sigma_\delta^2(\sigma_\delta^2\lambda_B^2 + \sigma_f^2)} - \lambda_B\sigma_\delta^2 \\ d\widehat{W}_{Bt}^\delta &= \frac{1}{\sigma_\delta} \left(\frac{d\delta_t}{\delta_t} - \widehat{f}_{Bt}dt \right).\end{aligned}$$

A.3 Proof of Proposition 1

The dynamics of \widehat{f}_A under the measure \mathbb{P}^B are written

$$\begin{aligned}d\widehat{f}_{At} &= \lambda_A(\bar{f} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta^2}(\widehat{f}_{Bt} - \widehat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}_{Bt}^\delta \\ &= \lambda_A\bar{f}dt + \lambda_A\widehat{g}_tdt - \lambda_A\widehat{f}_{Bt}dt + \frac{\gamma_A}{\sigma_\delta^2}\widehat{g}_tdt + \frac{\gamma_A}{\sigma_\delta}d\widehat{W}_{Bt}^\delta\end{aligned}$$

because by Girsanov's Theorem

$$d\widehat{W}_{At}^\delta = d\widehat{W}_{Bt}^\delta + \frac{1}{\sigma_\delta}\widehat{g}_tdt.$$

Consequently, the dynamics of \widehat{g} satisfy

$$\begin{aligned} d\widehat{g}_t &\equiv d\widehat{f}_{Bt} - d\widehat{f}_{At} \\ &= \left[(\lambda_A - \lambda_B)(\widehat{f}_{Bt} - \bar{f}) - \left(\lambda_A + \frac{\gamma_A}{\sigma_\delta^2} \right) \widehat{g}_t \right] dt + \frac{\gamma_B - \gamma_A}{\sigma_\delta} d\widehat{W}_{Bt}^\delta. \end{aligned}$$

A.4 Proof of Proposition 3

The optimization problem for agent B is

$$\begin{aligned} &\max_{c_{Bt}} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \frac{c_{Bt}^{1-\alpha}}{1-\alpha} dt \right] \\ \text{s.t. } &\mathbb{E} \left[\int_0^\infty \xi_t^B c_{Bt} dt \right] \leq x_{B0}, \end{aligned}$$

where ξ^B denotes the state-price density perceived by agent B and x_{B0} is his or her initial wealth. The problem for agent A (under the probability measure \mathbb{P}^B) is

$$\begin{aligned} &\max_{c_{At}} \mathbb{E} \left[\int_0^\infty \eta_t e^{-\rho t} \frac{c_{At}^{1-\alpha}}{1-\alpha} dt \right] \\ \text{s.t. } &\mathbb{E} \left[\int_0^\infty \xi_t^B c_{At} dt \right] \leq x_{A0}. \end{aligned} \tag{21}$$

Note how the change of measure enters the objective function of agent A , but that the expectation in the budget constraint (21) does not need to be adjusted. This is because the state-price density inside the expectation, ξ^B , is the one perceived by agent B .²³

The first-order conditions are

$$\begin{aligned} c_{Bt} &= (\kappa_B e^{\rho t} \xi^B)^{-\frac{1}{\alpha}} \\ c_{At} &= \left(\frac{\kappa_A}{\eta_t} e^{\rho t} \xi^B \right)^{-\frac{1}{\alpha}}, \end{aligned}$$

where κ_A and κ_B are the Lagrange multipliers associated with the budget constraints of agents A and B . Summing up the agents' optimal consumption policies and imposing market clearing, i.e., $c_{At} + c_{Bt} = \delta_t$, yields the state-price density perceived by agent B :

$$\xi_t^B = e^{-\rho t} \delta_t^{-\alpha} \left[\left(\frac{\eta_t}{\kappa_A} \right)^{1/\alpha} + \left(\frac{1}{\kappa_B} \right)^{1/\alpha} \right]^\alpha$$

Substituting the state-price density ξ^B in the optimal consumption policies yields the following consumption sharing rules

$$\begin{aligned} c_{At} &= \omega(\eta_t) \delta_t \\ c_{Bt} &= [1 - \omega(\eta_t)] \delta_t, \end{aligned}$$

²³Alternatively, we could have defined ξ^A , the state-price density under agent A 's probability measure. Then, we would have $\mathbb{E}^A [\xi^A \mathbf{1}_x] = \mathbb{E}^B [\eta \xi^A \mathbf{1}_x] = \mathbb{E}^B [\xi^B \mathbf{1}_x]$ for any event x . This implies that $\xi^B = \eta \xi^A$.

where $\omega(\eta)$ denotes agent A 's share of consumption, which satisfies

$$\omega(\eta_t) = \frac{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}}{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} + \left(\frac{1}{\kappa_B}\right)^{1/\alpha}}.$$

We assume, as in Yan (2008) and Dumas, Kurshev, and Uppal (2009), that the relative risk aversion α is an integer. The state-price density at time T satisfies

$$\begin{aligned} \xi_T^B &= e^{-\rho T} \delta_T^{-\alpha} \left(\left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_T}{\kappa_A}\right)^{1/\alpha} \right)^\alpha \\ &= e^{-\rho T} \delta_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{\eta_T \kappa_B}{\kappa_A}\right)^{\frac{j}{\alpha}} \\ &= e^{-\rho T} \delta_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}} \left(\frac{\eta_t \kappa_B}{\kappa_A}\right)^{\frac{j}{\alpha}} \eta_T^{\frac{j}{\alpha}} \\ &= e^{-\rho T} \delta_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}} \left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)}\right)^j \eta_T^{\frac{j}{\alpha}}, \end{aligned} \tag{22}$$

where the last equality comes from the fact that

$$\begin{aligned} \omega(\eta_t) &= \frac{\left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}}{\left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}} \\ 1 - \omega(\eta_t) &= \frac{\left(\frac{1}{\kappa_B}\right)^{1/\alpha}}{\left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}} \end{aligned} \tag{23}$$

and consequently

$$\left(\frac{\eta_t \kappa_B}{\kappa_A}\right)^{\frac{1}{\alpha}} = \frac{\omega(\eta_t)}{1 - \omega(\eta_t)}.$$

Rewriting Equation (23) yields

$$\left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} = \left(\frac{1}{1 - \omega(\eta_t)}\right)^\alpha \frac{1}{\kappa_B}. \tag{24}$$

Thus the single-dividend paying stock price satisfies

$$\begin{aligned}
S_t^T &= \mathbb{E}_t \left(\frac{\xi_T^B}{\xi_t^B} \delta_T \right) \\
&\stackrel{(22) \text{ and } (24)}{=} \mathbb{E}_t \left(\frac{e^{-\rho T} \delta_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{1}{\eta_t} \right)^{\frac{j}{\alpha}} \left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)} \right)^j \eta_T^{\frac{j}{\alpha}}}{e^{-\rho t} \left(\frac{1}{1-\omega(\eta_t)} \right)^{\alpha} \frac{1}{\kappa_B} \delta_t^{-\alpha}} \delta_T \right) \\
&= \mathbb{E}_t \left(e^{-\rho(T-t)} \frac{\sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{1}{\eta_t} \right)^{\frac{j}{\alpha}} \left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)} \right)^j \eta_T^{\frac{j}{\alpha}}}{\left(\frac{1}{1-\omega(\eta_t)} \right)^{\alpha} \delta_t^{-\alpha}} \delta_T^{1-\alpha} \right) \\
&= e^{-\rho(T-t)} (1-\omega(\eta_t))^{\alpha} \delta_t^{\alpha} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left(\frac{1}{\eta_t} \right)^{\frac{j}{\alpha}} \left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)} \right)^j \mathbb{E}_t \left(\eta_T^{\frac{j}{\alpha}} \delta_T^{1-\alpha} \right).
\end{aligned}$$

Finally the stock price is given by

$$S_t = \int_t^{\infty} S_t^u du.$$

The wealth of agent B at time t satisfies

$$V_{Bt} = \mathbb{E}_t \left(\int_t^{\infty} \frac{\xi_u^B}{\xi_t^B} c_{Bu} du \right).$$

The definitions of agent B 's consumption, c_B , the state-price density, ξ^B , and the share of consumption, $\omega(\eta)$, imply that

$$\begin{aligned}
\mathbb{E}_t \left(\frac{\xi_u^B}{\xi_t^B} c_{Bu} \right) &= \mathbb{E}_t \left(e^{-\rho(u-t)} \frac{\left[\left(\frac{1}{\kappa_B} \right)^{1/\alpha} + \left(\frac{\eta_u}{\kappa_A} \right)^{1/\alpha} \right]^{\alpha}}{\left[\left(\frac{1}{\kappa_B} \right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A} \right)^{1/\alpha} \right]^{\alpha} \delta_t^{-\alpha}} (1-\omega(\eta_u)) \delta_u^{1-\alpha}} \right) \\
&= \mathbb{E}_t \left(e^{-\rho(u-t)} \kappa_B (1-\omega(\eta_t))^{\alpha} \delta_t^{\alpha} \left(\frac{1}{\kappa_B} \right)^{1/\alpha} \left[\left(\frac{1}{\kappa_B} \right)^{1/\alpha} + \left(\frac{\eta_u}{\kappa_A} \right)^{1/\alpha} \right]^{\alpha-1} \delta_u^{1-\alpha}} \right) \quad (25)
\end{aligned}$$

Since the relative risk aversion α is an integer we have

$$\begin{aligned}
\left[\left(\frac{1}{\kappa_B} \right)^{1/\alpha} + \left(\frac{\eta_u}{\kappa_A} \right)^{1/\alpha} \right]^{\alpha-1} \left(\frac{1}{\kappa_B} \right)^{1/\alpha} &= \left[\left(\frac{\eta_u \kappa_B}{\kappa_A} \right)^{1/\alpha} + 1 \right]^{\alpha-1} \frac{1}{\kappa_B} \\
&= \frac{1}{\kappa_B} \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} \left(\frac{\eta_u \kappa_B}{\kappa_A} \right)^{j/\alpha} \\
&= \frac{1}{\kappa_B} \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} \left(\frac{1}{\eta_t} \right)^{j/\alpha} \left(\frac{\eta_t \kappa_B}{\kappa_A} \right)^{j/\alpha} \eta_u^{j/\alpha} \\
&\stackrel{(A.4)}{=} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} \left(\frac{1}{\eta_t} \right)^{j/\alpha} \left(\frac{\omega(\eta_t)}{1-\omega(\eta_t)} \right)^j \eta_u^{j/\alpha} \quad (26)
\end{aligned}$$

Substituting Equation (26) in Equation (25) yields the desired result

$$\mathbb{E}_t \left(\frac{\xi_u^B}{\xi_t^B} c_{Bu} \right) = e^{-\rho(u-t)} \delta_t^\alpha \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} \omega(\eta_t)^j (1-\omega(\eta_t))^{\alpha-j} \mathbb{E}_t \left(\left(\frac{\eta_u}{\eta_t} \right)^{j/\alpha} \delta_u^{1-\alpha} \right).$$

A.5 State Vector and Transform Analysis

Finding the equilibrium price boils down to computing the following expectation:

$$\mathbb{E}_t \left[\eta_u^{\frac{j}{\alpha}} \delta_u^{1-\alpha} \right] = \mathbb{E}_t \left[e^{(1-\alpha \ 0 \ 0 \ j/\alpha \ 0 \ 0 \ 0)X} \right], \quad (27)$$

where we define the augmented vector of state variables X by

$$X = \left(\zeta \quad \widehat{f}_B \quad \widehat{g} \quad \mu \quad \widehat{g}^2 \quad \widehat{g}\widehat{f}_B \quad \widehat{f}_B^2 \right)^\top. \quad (28)$$

In Equation (28), ζ represents the log aggregate consumption ($\zeta \equiv \ln \delta$), whereas μ represents the log relative outlook ($\mu \equiv \ln \eta$). Observe that the vector of state variables (initially four) has been augmented by adding three quadratic and cross-product terms. By doing so, the initially *affine-quadratic* vector $(\zeta, \widehat{f}_B, \widehat{g}, \mu)^\top$ is transformed into the *affine* vector X (see Cheng and Scaillet, 2007). It follows that the expectation in Equation (27) is the moment-generating function of an affine vector and thus we can apply the theory of affine processes (Duffie, Pan, and Singleton, 2000) to compute this quantity, which becomes

$$\mathbb{E}_t \left[\eta_u^{\frac{j}{\alpha}} \delta_u^{1-\alpha} \right] = \mathbb{E}_t \left[e^{(1-\alpha \ 0 \ 0 \ j/\alpha \ 0 \ 0 \ 0)X} \right] = e^{\widetilde{\alpha}(u-t) + \widetilde{\beta}(u-t)X_t}. \quad (29)$$

In Equation (29), $\widetilde{\alpha}$ is a 1-dimensional function of the maturity u , with boundary condition $\widetilde{\alpha}(0) = 0$, whereas $\widetilde{\beta}$ is a 7-dimensional function of the maturity u , with boundary condition $\widetilde{\beta}(0) = (1 - \alpha \ 0 \ 0 \ j/\alpha \ 0 \ 0 \ 0)$.

In order to solve Equation (29), let us write the dynamics of the affine state-vector X as follows:

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)d\widehat{W}_{Bt}^\delta \\ \mu(X) &= K_0 + K_1X \\ \left(\sigma(X)\sigma(X)^\top\right)_{ij} &= H_{0ij} + H_{1ij} \cdot X. \end{aligned}$$

From Duffie (2010) we know that

$$\mathbb{E}_t(\delta_u^\epsilon \eta_u^\chi) = \mathbb{E}_t\left(e^{\epsilon\zeta_u + \chi\mu_u}\right) = e^{\tilde{\alpha}(\tau) + \tilde{\beta}(\tau)X_t},$$

where $\tau = u - t$ and ϵ and χ are arbitrary constants. $\tilde{\alpha}$ and $\tilde{\beta}$ solve the following system of 8 Riccati ODEs

$$\begin{aligned} \tilde{\beta}'(\tau) &= K_1^\top \tilde{\beta}(\tau) + \frac{1}{2} \tilde{\beta}^\top(\tau) H_1 \tilde{\beta}(\tau) \\ \tilde{\alpha}'(\tau) &= K_0^\top \tilde{\alpha}(\tau) + \frac{1}{2} \tilde{\beta}^\top(\tau) H_0 \tilde{\beta}(\tau) \end{aligned}$$

with boundary conditions $\tilde{\beta}_1(0) = \epsilon$, $\tilde{\beta}_2(0) = 0$, $\tilde{\beta}_3(0) = 0$, $\tilde{\beta}_4(0) = \chi$, $\tilde{\beta}_5(0) = 0$, $\tilde{\beta}_6(0) = 0$, $\tilde{\beta}_7(0) = 0$, and $\alpha(0) = 0$. This system cannot be directly solved in closed form. However, we know that $\tilde{\beta}_1(\tau) = \epsilon$ and $\tilde{\beta}_4(\tau) = \chi$. Thus, the system can be written in a matrix Riccati form as follows²⁴

$$Z' = J + B^\top Z + ZB + ZQZ,$$

where

$$Z = \begin{pmatrix} \Gamma & \tilde{\beta}_3/2 & \tilde{\beta}_2/2 \\ \tilde{\beta}_3/2 & \tilde{\beta}_5 & \tilde{\beta}_6/2 \\ \tilde{\beta}_2/2 & \tilde{\beta}_6/2 & \tilde{\beta}_7 \end{pmatrix}$$

and Γ is a function of τ . The matrices J , B , and Q satisfy

$$\begin{aligned} J &= \begin{pmatrix} 0 & -\frac{\epsilon\chi}{2} & \frac{\epsilon}{2} \\ -\frac{\epsilon\chi}{2} & \frac{(\chi-1)\chi}{2\sigma\delta^2} & 0 \\ \frac{\epsilon}{2} & 0 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 & 0 \\ -\gamma_A\epsilon + \gamma_B\epsilon - (\lambda_A - \lambda_B)\bar{f} & -\frac{\lambda_A\sigma\delta^2 + \gamma_A - \gamma_{AX} + \gamma_{BX}}{\sigma\delta^2} & \lambda_A - \lambda_B \\ \gamma_B\epsilon + \bar{f}\lambda_B & -\frac{\gamma_{BX}}{\sigma\delta^2} & -\lambda_B \end{pmatrix} \\ Q &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2(\gamma_A - \gamma_B)^2}{\sigma\delta^2} & \frac{2\gamma_B(\gamma_B - \gamma_A)}{\sigma\delta^2} \\ 0 & \frac{2\gamma_B(\gamma_B - \gamma_A)}{\sigma\delta^2} & \frac{2\gamma_B^2}{\sigma\delta^2} \end{pmatrix}. \end{aligned}$$

Note that we set J_{11} and J_{23} to zero since they can be any real numbers. Using Radon's lemma,

²⁴See Andrei and Cujean (2010) for detailed explanations related to this methodology.

we get

$$\begin{aligned} Z(\tau) &= Y^{-1}(\tau)X(\tau) \text{ where } X \text{ and } Y \text{ satisfy} \\ X' &= BX + JY, \quad X(0) = [0]_{3 \times 3} \\ Y' &= -QX - B^\top Y, \quad Y(0) = I_{3 \times 3}. \end{aligned}$$

The solution of this system is

$$\begin{aligned} (X(\tau) \ Y(\tau)) &= (X(0) \ Y(0)) M(\tau), \text{ where } M(\tau) \text{ is the matrix exponential} \\ M(\tau) &= \exp\left(\begin{pmatrix} B & -Q \\ J & -B^\top \end{pmatrix} \tau\right). \end{aligned}$$

Note that the matrix exponential $M(\tau)$ has to be computed using a Jordan decomposition. Indeed, we have

$$M(\tau) = S \exp(J_o \tau) S^{-1},$$

where J_o and S are, respectively, the Jordan and the similarity matrix extracted from the Jordan decomposition. The Betas are consequently given by

$$\begin{aligned} \tilde{\beta}_1(\tau) &= \epsilon \\ \tilde{\beta}_2(\tau) &= \frac{n_{01} + \sum_{i=1}^8 n_{i1} e^{j_i \tau}}{b_{01} + \sum_{i=1}^8 b_{i1} e^{j_i \tau}} \\ \tilde{\beta}_3(\tau) &= \frac{n_{02} + \sum_{i=1}^8 n_{i2} e^{j_i \tau}}{b_{02} + \sum_{i=1}^8 b_{i2} e^{j_i \tau}} \\ \tilde{\beta}_4(\tau) &= \chi \\ \tilde{\beta}_5(\tau) &= \frac{n_{03} + \sum_{i=1}^8 n_{i3} e^{j_i \tau}}{b_{03} + \sum_{i=1}^8 b_{i3} e^{j_i \tau}} \\ \tilde{\beta}_6(\tau) &= \frac{n_{04} + \sum_{i=1}^8 n_{i4} e^{j_i \tau}}{b_{04} + \sum_{i=1}^8 b_{i4} e^{j_i \tau}} \\ \tilde{\beta}_7(\tau) &= \frac{n_{05} + \sum_{i=1}^8 n_{i5} e^{j_i \tau}}{b_{05} + \sum_{i=1}^8 b_{i5} e^{j_i \tau}}. \end{aligned}$$

Notice that the function $\tilde{\alpha}(\tau)$ is obtained through a numerical integration. Thus, this function is not obtained in closed form. Since in our setup $\chi = \frac{j}{\alpha}$ and $\epsilon = 1 - \alpha$, the stock price simplifies to

$$\begin{aligned} S_t &= \int_0^\infty S_t^\tau d\tau \\ &= \delta_t \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega(\eta_t)^j (1 - \omega(\eta_t))^{\alpha-j} \times \\ &\quad \times \int_0^\infty e^{-\rho\tau} e^{\tilde{\alpha}_j(\tau) + \tilde{\beta}_{2j}(\tau) \hat{f}_{Bt} + \tilde{\beta}_{3j}(\tau) \hat{g}_t + \tilde{\beta}_{5j}(\tau) \hat{g}_t^2 + \tilde{\beta}_{6j}(\tau) \hat{f}_{Bt} \hat{g}_t + \tilde{\beta}_{7j}(\tau) \hat{f}_{Bt}^2} d\tau. \end{aligned}$$

Even though the above integral is computed numerically, the price process can be simulated very efficiently.

A.6 Robustness Analysis

Table 7 confirms that disagreement and long-run risk have different impacts on stock-return volatility. Indeed, an increase in long-run risk increases the average level of volatility (Bansal and Yaron, 2004), while an increase in disagreement increases both the variation of volatility and the persistence of volatility.

Economy	Parameter	Mean	Min.	Max.	Vol.	Persist.
(1) No model disagreement and severe long-run risk	$\lambda_A = 0.1$ $\lambda_B = 0.1$	0.169	0.164	0.181	0.002	0.463
(2) Moderate model disagreement and strong long-run risk	$\lambda_A = 0.2$ $\lambda_B = 0.1$	0.101	0.081	0.147	0.012	0.913
(3) Severe model disagreement and moderate long-run risk	$\lambda_A = 0.3$ $\lambda_B = 0.1$	0.085	0.046	0.199	0.028	0.997
(4) Severe model disagreement and weak long-run risk	$\lambda_A = 0.4$ $\lambda_B = 0.2$	0.023	0.013	0.043	0.005	0.996
(5) Severe model disagreement and no long-run risk	$\lambda_A = 0.5$ $\lambda_B = 0.3$	0.005	0.002	0.017	0.003	0.999

Table 7: Properties of volatility for various calibrations

This table presents the mean, minimum, maximum, volatility, and persistence of volatility in five different models. Persistence is calculated as the sum of the parameters β_1 and β_2 in the GARCH(1,1) estimation. In bold is the benchmark model considered throughout the paper. Numbers reported above are (annualized) averages computed over 1,000 simulations of weekly data over a 100-year horizon.

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